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THE MATHEMATICAL REPOSITORY.

VOL. II.

CONTAINING

Algebraical Solutions

OF

A great Number of PROBLEMS,

In ſeveral Branches of the

MATHEMATICS.

I. Indetermined Queſtions, ſolved generally, by an elegant Method communicated by Mr. *De Moivre*.

II. Many curious Queſtions relating to Chances and Lotteries.

III. A great Number of Queſtions concerning Annuities for Lives, and their Reverſions; wherein that Doctrine is illuſtrated in a Multitude of intereſting Caſes, with numeral Examples, and Rules in Words at length, for thoſe who are unacquainted with the Elements of theſe Sciences, *&c.*

By *JAMES DODSON*,
Accomptant, and Teacher of the MATHEMATICS.

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MDCCLIII.

G.III
BR. MUS.

TO

David Papillon, Eſq; F.R.S.

One of the Honourable Commiſſioners of the Exciſe.

SIR,

ALthough the valuation of annuities on lives, the ſubject principally treated of in this book, is of great importance in moſt parts of the known world, and particularly in the *Britiſh* empire; yet it is a branch of mathematical learning, which has been but lately cultivated, and probably not yet brought to that perfection which it is capable of.

I have ventured to publiſh the reſult of my endeavours to facilitate this kind of intereſting and difficult computations, becauſe I conceiv'd that I had made ſome improvement therein; and I have been encouraged to addreſs it to you, becauſe you are known to be a gentleman, who is, not only, well acquainted with the mathematical ſciences, but alſo, an encourager of thoſe who attempt to promote uſeful learning.

Should you honour this work with a peruſal, I hope, SIR, you will excuſe, or at leaſt paſs a favourable cenſure upon, the imperfections which your great knowledge, in thoſe things, may diſcover: Theſe, I hope, will be few, becauſe I have endeavoured (to the utmoſt of my ability) to preſent you with a treatiſe, that might be worthy of your acceptance and approbation.

I am,

SIR,

Your moſt obliged,

And moſt humble Servant,

JAMES DODSON.

THE

PREFACE.

THE general method of ſolving indetermined queſtions, which is introduced at the beginning of this volume, was compoſed by the celebrated mathematician, whoſe name it bears, ſome years ago, and was then conceived in the form of a letter, directed to William Jones, *Eſq; with a deſign that the ſame ſhould have been printed in the philoſophical tranſactions, in that form; but Mr.* Jones's *death having prevented that, it was generouſly communicated to the author, to be inſerted in this work, in ſuch a form as would be conformable thereto.*

This communication determined the author to reaſſume the ſubject of indetermined queſtions, which, though it bears a place in the firſt volume, is not there handled, either ſo generally, or ſo elegantly, as here.

Mr. De Moivre *propoſed to ſhew the manner of ſolving thoſe queſtions, in which there are three unknown quantities and but one equation; and in order to render that the more intelligible, he introduced one example, in which there were but two unknown quan-*

tities, to which the author hath added a few more, to make that method of ſolution familiar to the reader.

Mr. De Moivre's ſolution of equations, containing three unknown quantities, is truly elegant, and equal to his other performances; and, as the author found that thoſe methods might be extended to equations in which there are four unknown quantities, he has endeavoured to ſhew how it may be done; but leſt any inelegance or inaccuracy of his ſhould, for want of a proper diſtinction, be imputed to Mr. De Moivre, he has remarked the place where that gentleman's performance ends.

In the ſecond part of the firſt volume of this work ſeries of different kinds are treated of in a general manner, the application of many of which, to real uſe, is not therein contained; it was therefore thought convenient to reſume that ſubject alſo, and, for the ſake of ſtudents in algebra, to ſhew in what manner they are actually conducive to the facilitating the computations of many queſtions, which frequently occur in the tranſactions of mankind.

And here it was impoſſible to paſs over their uſefulneſs in a ſcience, which (although at all times neceſſary to be underſtood) was altogether unknown to the antient mathematicians, and has been but very lately brought to any degree of perfection; the intelligent reader will eaſily perceive, that the doctrine of chances is here meant; upon which ſubject there are hitherto but few writers in any language, and Mr. De Moivre, the author of the firſt ſyſtem thereof in Engliſh, is yet alive.

It will therefore be very eaſy for the reader to ſee the reaſon, why there is not ſuch a pompous liſt of authors quoted in this preface, as in that to the former volume; for when Huygens, Monmort, *and* Bernoulli; De Moivre, *and* Simpſon, *have been named, the other writers, on the ſubject, muſt be ſearched for among the tranſactions of the ſeveral royal ſocieties, and other miſcellaneous and periodical works.*

The author found himſelf obliged to introduce the ſolutions of ſeveral queſtions, relating to this ſcience, which do not depend on ſeries, in order to enable the learner to underſtand the reaſons of thoſe ſolutions which do; and in theſe, becauſe they contain as it were the principles of the ſcience, he has been obliged to follow one, or more, of the above-named authors ſtrictly; but in the ſolutions of queſtions, which are not ſo fundamental, he has frequently aimed at an improvement, and hopes with ſome ſucceſs.

But of all the various kinds of problems relating to chance, there are none ſo intereſting, to the inhabitants of theſe kingdoms, as thoſe relating to annuities for lives, and the reverſions of them; which will be evident when we conſider the great property veſted in them.

The preſent poſſeſſors of entailed eſtates, are in the common law juſtly called tenants for life; marriage ſettlements generally convey the reverſion of a conſiderable part of the bridegroom's eſtate, to the bride, *for her natural life after his deceaſe; to which two things all the freehold eſtates in theſe kingdoms are liable; and if to theſe be added, the great number of copyholds determinable on lives; the great quantities of church, college, and other lands leaſed on lives; and the eſtates poſſeſſed by eccleſiaſtical perſons of all degrees, we ſhall find, that the values of the poſſeſſions, and the reverſions, of much the greateſt part of the real eſtates, in theſe kingdoms, will, one way or other, depend on the values of lives.*

Likewiſe the incomes annexed to all places, civil and military, all penſions, and moſt charitable donations, are annuities for life; the intereſts or dividends of many perſonalities in the ſtocks have been, by the wills of their poſſeſſors, rendered of the ſame kind; beſides which there are ſome annuities on lives which have been granted by the government, and have parliamentary ſecurity for their payment; and others that have been granted by pariſhes, and other communities, in conſequence of acts of parliament made for that purpoſe.

To the ſolution of queſtions of this ſort, therefore, the author hath, in this volume, applied the ſummation of thoſe kind of ſeries which the reader will find, in queſtions 185 *to* 193, *part* II. *vol.* I. *having firſt, in queſtions* 15 *to* 20 *hereof, for the ſake of perſpicuity, repreſented and ſummed ſuch particular ſeries as could be applied to this purpoſe, in a manner ſomewhat different from thoſe above quoted.*

Theſe preparatory queſtions are placed directly after thoſe of the indetermined kind, becauſe the chain of thoſe relating to chance ſhould be uninterrupted; and are followed by queſtions 21 *and* 22, *which (till their application, to the approximation of the values of joint lives, appears in queſtions* 64 *and* 69) *ſeem to have no relation to the reſt of the work; and may, therefore, if the reader chooſes, remain unread till wanted.*

The ſolutions of the following queſtions, to the 29*th, exhibit the firſt principles of the doctrine of chances; and though the words of each of theſe queſtions ſtate only a particular caſe, they are of general application.*

The ſubſequent queſtions are claſſed, and contain the following probabilities, viz.

The 30*th and* 31*ſt, thoſe of the happening of one event, or more, in* 2, 3, *or* m *trials;*

The 32*d,* 33*d, and* 34*th, thoſe of one event and no more in* 2, 3, 4, *or* m *trials;*

The 35*th and* 36*th, thoſe of two events, or more, in* 3, 4, *or* m *trials;*

And, the 37*th,* 38*th, and* 39*th, thoſe of two events, and no more, in* 3, 4, 5, *or* m *trials.*

The queſtions, from 40 *to* 44, *both incluſive, apply this doctrine to the play at Backgammon.*

The ſolutions from thence, to queſtion 48, *are introduced to ſhew, that all queſtions, that can poſſibly be aſked, concerning the happening, or failing of any number of events, in* 2, 3, 4, *or* m *trials, may be anſwered by one, or more, of the terms, of the* 2*d*, 3*d*, 4*th*, *or* m*th power, of that binomial, whoſe root is the ſum of the chances, for the happening and failing of one ſuch event.*

The queſtions, numbered 49 *to* 54, *relate to lotteries, and are introduced for the ſake of the general ſolution, given in queſtion* 55, *for determining how many tickets ought to be purchaſed, to procure an equal chance for the obtaining, at leaſt, of* 1, 2, 3, 4, *or* p, *prizes.*

Previous to our ſpeaking of the queſtions relating to annuities on lives, and their reverſions, it may be convenient to obſerve, that the antient ways of determining theſe values depend upon different cuſtoms, which ſeem to have been eſtabliſhed, in the places where they are uſed, merely for want of good methods of calculation, which cuſtoms are ſtill in uſe in ſome places; but as the advantages that will attend the determination of theſe things, by computation, preferably to theſe cuſtoms, are obvious; it may ſeem ſtrange, that (notwithſtanding many of theſe tenures have ſubſiſted from the very origin of private property in theſe kingdoms, yet) we do not meet with ſo much as an attempt towards computing their values, till that of the late juſtly celebrated Dr. Halley, *by the aſſiſtance of the* *bills of mortality at* Breſlaw *in* Sileſia; *which was ſoon followed by Mr.* De Moivre'*s truly admirable hypotheſis, that the decrements of life may be eſteemed nearly equal, after a certain age.*

It has been the opinion of ſome authors, that ſince this hypotheſis was originally derived from the Breſlaw *obſervations, it cannot be near ſo well adapted to the inhabitants of theſe kingdoms, as what has been deriv'd from the bills of mortality of* London; *but this argument doth not (as the author conceives) appear to be concluſive.*

Firſt, becauſe thoſe bills, as hitherto kept, are not well adapted to anſwer this purpoſe.

Secondly, becauſe the manner in which the inhabitants of London, *and thoſe of moſt of the country towns and villages, live; their occupations, diet, and diverſions; nay, the very air they breathe, are as different, as thoſe of* London *and* Breſlaw *can poſſibly be; and conſequently, ſo muſt the times of their diſſolution; all which has been, with a great deal of clearneſs, evidenced by Mr.* Corbyn Morris *in a pamphlet, called,* Obſervations on the paſt growth and preſent ſtate of London.

Thirdly, becauſe thoſe perſons, who ſuppoſe that Mr. De Moivre'*s hypotheſis has its foundation, particularly, in the* Breſlaw *obſervations, are greatly miſtaken; for, on the contrary, if the* London *obſervations*

tions had been then in Mr. De Moivre*'s hands, he might, as juſtly, have derived his hypotheſis from them; which will appear from his own words, in the preface to his treatiſe of annuities on lives, compared with the* London *obſervations.*

“ Two or three years after the publication of my doctrine of chances (*ſays that excellent mathematician*) I took the ſubject into conſideration; and conſulting Dr. *Halley*'s table of obſervations (*ſee page* 149 *of this work*) I found that the decrements of life, for conſiderable intervals of time, were in arithmetic progreſſion; for inſtance, out of 646 perſons of twelve years of age, there remain 640 after one year, 634 after two years, 628, 622, 616, 610, 604, 598, 592, 586, after 3, 4, 5, 6, 7, 8, 9, 10, years reſpectively, the common difference of thoſe numbers being 6.

“ Examining afterwards other caſes, I found that the decrements of life, for ſeveral years, were ſtill in arithmetic progreſſion; which may be obſerved from the age of 54 to the age of 71, where the difference for 17 years together, is conſtantly 10.

“ After having thoroughly examined the tables of obſervations, and diſcovered that property of the decrements of life, I was inclined to compoſe a table of the values of annuities on lives, by "keeping cloſe to the table of obſervations; which "would have been done with eaſe, by taking, in "the whole extent of life, ſeveral intervals, whe"ther equal or unequal; however, before I under"took the taſk, I tried what would be the reſult "of ſuppoſing thoſe decrements uniform, from the "age of twelve; being ſatisfied that the exceſſes "ariſing on one ſide, would be nearly compen"ſated by the defects on the other; then compar"ing my calculation with that of Dr. *Halley*, I "found the concluſion ſo very little different, that "I thought it ſuperfluous to join together ſeveral "different rules, in order to compoſe one."

Now the ſame thing, which Mr. De Moivre *mentions above, happens in the tables of the* London *obſervations (*ſee page 157 of this work) viz. *out of* 510 *perſons of twelve years of age, there remain* 504 *after one year,* 498 *after two years,* 492, 486, 480, 474, 468, 462, *after* 3, 4 5, 6, 7, *and* 8 *years reſpectively, the common difference being* 6; *and the like happens in many other inſtances; but the lengths of the intervals differ from thoſe in the* Breſlaw *tables.*

Now, ſince either of thoſe tables of obſervations might have furniſhed the ſagacious Hypotheſiſt *with the invention thereof; and ſince there is no reaſon to doubt, but that the bills of mortality of other places will furniſh tables having the ſame general properties (although*

(although the lengths of the intervals, and manner of the increaſe, or decreaſe, of their differences may not be the ſame with either of theſe); therefore it is highly probable, that if the obſervations, drawn from the bills of mortality of a great number of places, were added together, and a mean table compoſed therefrom, that the numbers, therein contained, would be, at leaſt for larger intervals than in either of theſe, truly arithmetical.

And if this ſhould prove ſo, the hypotheſis may be ſaid to be deduced from the bills of mortality of the world; and will be much more generally uſeful, than any particular table of obſervations.

However, if the argument for the uſe of the London *obſervations be admitted, we ſhall want ſuch tables for every place, wherein a perſon, whoſe life is to be valued, may uſually reſide, in order to be able to calculate it to a ſufficient exactneſs; and theſe, indeed, it is to be wiſhed, were actually in being* *; *and, whenever ſuch tables can be obtained, a method of calculating therefrom is provided in this work.*

In the mean time, we muſt have recourſe to the hypotheſis, for the calculations of ſuch lives as are not reſident in London, *and a few more great cities.*

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* See the author's letter upon this ſubject, page 333, vol. 47, of the philoſophical tranſactions.

Suppoſing, therefore, the decrements of life to be equal, the ſolutions of all queſtions, that can be aſked, concerning annuities on one or more lives, and the reverſions of them, are here, for the ſtudents eaſe, anſwered by the application of the ſummations of the ſeries in queſtions 15 *to* 20; *although each of them might have been reduced to a different ſeries, and ſeparately ſolved, by the methods given in queſtions* 185 *to* 193, *part* II. *vol.* I.

The whole work of theſe queſtions is alſo, for the reader's eaſe, performed at length, without ſuppreſſing any of the intermediate ſteps; and the reſults of all thoſe operations, which it was conceived would be of frequent uſe, are (for the ſame reaſon) inſerted in words at length, after the example of Mr. De Moivre, *and are ſo contrived, that theſe rules ſeldom refer to thoſe going before, in order to diſcover the neceſſary method of operation.*

Farther, it has not been thought ſufficient, barely, to give a ſolution to theſe queſtions, but great pains have been taken (after an expreſſion of the anſwer has been obtained) to diſpoſe the parts thereof into that order, which appeared to give the eaſieſt numerical proceſs; and, as every ſtep neceſſary thereto is inſerted, the ſolutions are ('tis true) conſiderably lengthened; but then, the advantages that the operator will thereby receive, in his computation, and thoſe which the ſtudent

dent will obtain, in the management of his algebraic expreſſions, will ('tis preſumed) more than compenſate that diſadvantage.

An inſtance of this occurs in queſtion 56; *wherein the value of a ſingle life is inveſtigated, by Mr.* De Moivre*'s hypotheſis, the reſult whereof,* viz. $\frac{\overline{n-1+p}\times r-n}{n\times\overline{r-1}^{2}}$ *(in which* n *ſignifies the complement of life,* p *the preſent worth of one pound due at the end of that complement, and* r *the amount of one pound and its intereſt for one year) differs greatly, in appearance, from* $\frac{1-\frac{r}{n}P}{r-1}$, *the expreſſion given in his treatiſe of annuities,* n *and* r *having the ſame ſignification as before, and* P *ſignifying the preſent worth of an annuity certain, for as many years as are denoted by the complement of life.*

Now both of theſe expreſſions will give the ſame reſult, as will appear by the under-written operations of the example given in his treatiſe, viz. *the value of a life of* 50 *at five* per Cent. *where* n $=36$ *and* r$=1{,}05$.

Firſt, according to Mr. De Moivre.

Here P *(taken from the proper table) is* } 16,547;
Multiply by r, *inverted* = 50,1,

16,547
827

17,374

Divide by n = 36) 17,374 (,4826 *quotient.*

$\underline{144}$

297

$\underline{288}$

94

$\underline{72}$

22

The quotient ,4826, *being taken from unity, leaves* ,5174;

Divide by (1,05−1=),05) $\underline{,5174}$

10,35 *the value requir'd.*

Secondly, according to the reſult of queſtion 56,

p *(taken from the proper table) is* } ,1731, *and* n − 1 = 35;

Therefore n − 1 + p = 35,173;

Multiply by r, *inverted* $\underline{50,1,}$

35,173.

$\underline{1759}$

$\underline{36,932;}$

From which taking 36 remains } $\underline{,932}$ *the dividend.*

$\overline{r-1}^{2}=(,05\times,05=),0025;$

Multiply by $\underline{36.}$

150

$\underline{75}$

$\underline{,0900}$ *the diviſor:*

And ,09) ,932

,10,35 *the value requir'd;*

which ſecond operation is preferable to the former, in two reſpects.

Firſt, becauſe it can be performed by fewer figures; for in the firſt 52 *figures are abſolutely neceſſary, and in the ſecond but* 44.

Secondly, becauſe the tabular number, repreſented by p, *will be much eaſier obtained (if the proper tables are not at hand) than the tabular number* P; *for (ſince the one is only the preſent worth of one pound, and the other the preſent worth of an annuity of one pound for the ſame time) the latter cannot be found without firſt computing the former.*

The authors who have rejected Mr. De Moivre's *hypotheſis, and chooſe to make their computations from particular obſervations, have given us no methods of computing the values (even of ſingle lives) other, than what are particularly adapted to the two tables of obſervations above-mentioned, except that general one, of performing a multiplication and diviſion, for every year that the propoſed life can poſſibly continue in being.*

To remedy this defect Mr. De Moivre, *in a letter to* William Jones, *Eſq; (publiſhed in the philoſophical tranſactions No.* 473) *has given a method, deduced from fluxions, and requiring the uſe of an hyperbolic logarithm, for the finding the value of a life from a given table of obſervations; which method could have no place in this work, becauſe it is profeſſedly written for* *the uſe of thoſe who know nothing more than common algebra. The author, however, has been ſucceſsful enough to remove the obſtacle, and to furniſh the reader with a rule, which (conſidering the great difficulty of the problem) is very eaſy, requiring only thoſe principles which are neceſſary to the other queſtions, and a more frequent uſe of a table of the preſent worths of one pound.*

In inveſtigating the values of combined lives, the author has adhered more ſtrictly to the hypotheſis, than its great author himſelf has done; who, perceiving, perhaps, that the computations of the values of combined lives would, upon that principle, be more prolix than he could wiſh, has (in the contrivance of his rules) conſidered the decrements of life as partly in a conſtant ratio; whereby he obtains an eaſier operation, but at the ſame time confeſſes it to be but an approximation to the truth; for here, the reader will find all the aforementioned caſes of combined lives ſolved, in a manner ſtrictly conformable to the hypotheſis, as well as by Mr. De Moivre'*s approximation.*

But, becauſe the proceſſes, ariſiug from a ſtrict adherence to the hypotheſis, may be eſteemed too operoſe, and Mr. De Moivre'*s approximation has been objected to, as not ſufficiently accurate, the author has (by the aſſiſtance of queſtions* 21 *and* 22) *exhibited a new approximation to thoſe values, founded on the properties of arithmetical progreſſionals; the reſults of* *which do not (in all the caſes the author has yet had occaſion to try) differ from thoſe derived, ſtrictly, from the hypotheſis, by* $\frac{1}{10}$ *of a year's purchaſe, and require little more work (if any) than thoſe of* Mr. De Moivre.

It may be aſked, perhaps, why (after having diſcovered ſo eaſy and accurate an approximation to the values of joint lives) the author ſhould ſtill perſiſt in inveſtigating the true ſolutions to the ſubſequent queſtions, which take up a great deal of room, and generally give a numerical proceſs, tedious enough at beſt; when he might, either have directed thoſe anſwers to be found by the additions and ſubtractions of the values of joint lives (as has been done by former authors) or, at moſt, might have added and ſubtracted, only, the expreſſions of the approximations; which he has likewiſe done, in order to obtain thoſe rules which he has given in words at length?

To this queſtion, it may be properly replied, that it muſt certainly be very agreeable, to every lover of truth, to have it in his power to be as accurate in his computations as the nature of the ſubject will admit; that many things open themſelves to the reader's view, in the proſecution of thoſe ſolutions, that will be both entertaining and uſeful; for, not to mention the generality of the concluſion of each ſet of queſtions, it muſt, no doubt, give him a ſenſible pleaſure to find, that (by adhering ſtrictly to the hypotheſis as well as by the approximations)

mations) the value of the longeſt of three unequal lives may be found, independently of any previous operation, by a numerical proceſs, at leaſt as ſhort as that neceſſary to the obtaining the value of three unequal joint lives; which is only one, of the ſeven queſtions, that (according to the former authors) muſt be ſolved, previous to this; and it is reaſonable to ſuppoſe, that he will, from the experience of this, be armed with patience and reſolution enough, in any ſimilar caſe, to go through a long operation in hope of a like reſult: That to a reader of ſmall experience (for whom in particular this work is calculated) theſe ſolutions will be uſeful exerciſes of the theory of algebra, and render him ready at litteral computation: and laſtly, that, after all, the reſulting numerical operations are not ſo very operoſe, as to deter a perſon, who is uſed to figures, from computing the anſwers, truly, whenever the largeneſs of the property in queſtion, or the accuracy of the data, *ſhall render ſuch calculations neceſſary, or certain.*

The reader will therefore find, in the remaining part of the work, the values of annuities on two unequal joint lives, on 2, 3, 4, *or* m, *equal joint lives; on three unequal joint lives; and, on three joint lives, whereof two are equal, and the third either of a greater or leſſer age than thoſe: the values of annuities, to continue during the longeſt of any ſuch lives; the values of the reverſions of ſums certain, or eſtates, after ſingle, or any ſuch combinations of lives; the reverſi-*

ons of an annuity, on any life, after one life; after two equal, or unequal joint lives; or after the longeſt of two ſuch lives; alſo of two ſuch lives, after one: All of them ſolved, ſtrictly, from the hypotheſis, and alſo by the new approximation; in ſuch a manner, as will enable the reader to calculate the value of any ſuch life, or reverſion, without being previouſly obliged to refer to any of the beforegoing ſolutions.

The author has, indeed, inveſtigated the value of annuities, and reverſions, on unequal lives, and their approximations, no farther, than where three ſuch are concerned; as ſuppoſing thoſe will be ſufficient to anſwer all the purpoſes that will be commonly required; but then, when the lives are equal, the computation may be extended to any number of lives; and the numerical operations reſulting therefrom, adhering ſtrictly to the hypotheſis, are as eaſy as can be expected, in the ſolutions of queſtions of ſuch great difficulty; and, perhaps, cannot be ſhortened by any approximation.

Since the work of computing the values of combined lives, by the tables of obſervations, is exceedingly laborious; as appears by queſt. 68, *in which the value of two joint lives is computed from the* London *tables; a method of approximating, very nearly, to them, is, therefore, deduced from queſtion* 104; *by which the complement of life may be found, which, upon the ſuppoſition of equal decrements, will have the ſame probability of attaining the extremity of old age, as any*

any given life has, according to that table of obſervations by which its value might be calculated.

Whence, if the complements of any lives (ſo found) be ſubſtituted for the differences between the given ages and 86, *in any of the ſolutions before given, the reſult will be nearly equal to the anſwer, which would ariſe, by ſtrictly computing by ſuch tables of obſervations: To this queſtion is annexed a table of ſuch complements, adapted to the* London *obſervations, and ſome examples worked thereby.*

The queſtions following the 104*th, relate to the values of the expectations of lives; for finding which Mr.* De Moivre *has given rules, without their demonſtration, having only informed the reader, that he found the value of the expectation of a ſingle life by a calculation deduced from the method of fluxions: but here, the value of the expectation of a ſingle life is found, in a manner ſimilar to that of an annuity thereon; and the values of the expectations of combined lives are deduced therefrom, in the ſame manner as the approximations to the values of annuities on ſuch lives: and the numerical proceſs, for finding the expectation of the longeſt of any number of lives, is (as before) ſhorter, than that, for finding the expectation of any number of joint lives.*

The following tables are inſerted in this work, viz.

A table of the preſent worths of one pound, due at the end of any number of years, leſs than 101.

A table of the preſent values of annuities on ſingle lives, upon the ſuppoſition of equal decrements.

A table of the multiples of the ſixth part of the preſent worth of one pound, due at the end of one year; which is very uſeſul in the above mentioned new methods of approximating to the values of combined lives, and their reverſions.

All which tables are computed at the ſeveral rates of 3, 3½, 4, 4½, 5, *and* 6, per Cent.

As the deſign of this ſecond volume is the ſame with that of the firſt, viz. *the rendering of the more difficult parts of calculation eaſy, and familiar, to learners; and, as it has been endeavoured, in the execution thereof, to render it not altogether unworthy the peruſal of more experienced readers; the author hopes for the ſame favourable reception, which the firſt volume has met with.*

BELL-DOCK, WAPPING,
May 21, 1753.

THE
MATHEMATICAL
REPOSITORY.

The Solution *of indetermined Queſtions in Affirmative Integers, communicated by Mr.* Abraham De Moivre, *Fellow of the Royal Societies of* London *and* Berlin.

QUESTION I.

It is required to find two affirmative integers; the firſt of which being multiplied by 35, and the ſecond by 43, the ſum of thoſe products may be 4000?

Queſtions of this kind have been ſolved by ſeveral Mathematicians, this being inſerted here, only, as an introduction to thoſe that are more difficult.

SOLUTION.

By the queſtion, $35x + 43y = 4000,$

Or $35x = 4000 - 43y;$

Th. $x = \frac{4000 - 43y}{35}.$

Which fractional expreſſion is to be an integer, by the nature of the queſtion.

Now it is evident that, if from any integer, a leſſer integer be taken, the remainder will be an integer; therefore, in order to have a remainder in the ſimpleſt terms, let the greateſt integer poſſible, be taken from the above expreſſion.

Now $\frac{4000-43y}{35} = 114 - y + \frac{10-8y}{35}$; as will appear by actually dividing $4000 - 43y$ by 35.

If therefore $114 - y$ (the greateſt integer poſſible) be taken from the above expreſſion, the remainder muſt be an integer (ſuppoſe r) Then $r = \frac{10-8y}{35}$,

Or - - - $35r = 10 - 8y$,
Or - - - $8y = 10 - 35r$,
Therefore (dividing as before) $y = \left(\frac{10-35r}{8} = \right) 1 - 4r + \frac{2-3r}{8}$.

In which expreſſion $1 - 4r$ is the quotient, or greateſt integer,

And $\frac{2-3r}{8}$ the remainder; let $s = \frac{2-3r}{8}$;
Then - - - $8s = 2 - 3r$,
Or - - - - $3r = 2 - 8s$;
Therefore (dividing again) - $r = \left(\frac{2-8s}{3} = \right) -2s + \frac{2-2s}{3}$.

In which expreſſion $-2s$ is the quotient, or greateſt integer, and $\frac{2-2s}{3}$ remains let $t = \frac{2-2s}{3}$;
Then - - - $3t = 2 - 2s$,
Or - - - - $2s = 2 - 3t$;

Therefore (dividing once more) $s = 1 - t - \frac{t}{2}$;

From which laſt expreſſion it is evident, that $\frac{1}{2}t$ muſt be an integer; let therefore $p = \frac{1}{2}t$; Or $2p = t$;
Then by reſuming the former equation $s = \left(\frac{2 - 3 \times 2p}{2} = \right) 1 - 3p$;

And $r = \left(\frac{2-\overline{1-3p}\times 8}{3} = \right)\frac{2-8+24p}{3}$,

That is $r = \left(\frac{24p-6}{3} = \right) 8p-2$;

Again $y = \left(\frac{10-35\times\overline{8p-2}}{8} = \right)\frac{10-280p+70}{8}$,

That is $y = \left(\frac{80-280p}{8} = \right) 10-35p$;

Lastly $x = \frac{4000-43\times\overline{10-35p}}{35}$,

That is $x = \left(\frac{3570+43\times 35p}{35} = \right) 102+43p$.

Having thus got equations, expreſſing the values of x and y, in the terms of p; any integer may be aſſumed for p that will render x and y affirmative; but from the equation ($y = 10-35p$) it is evident, that p cannot be an affirmative integer; for, if it be, then y will be negative, which is contrary to the intent of the queſtion; p muſt therefore be, either 0, or a negative integer.

Now if $\begin{cases} p = 0; & x = 102+0 = 102; \text{ and } y = 10-0 = 10. \\ p = -1; & x = 102-43 = 59; \text{ and } y = 10+35 = 45. \\ p = -2; & x = 102-86 = 16; \text{ and } y = 10+70 = 80. \end{cases}$

It will appear very plain, that p cannot receive any more interpretations than the three before-going; if it be conſidered, that when $p = -3$, then $x = (102 - 3\times 43 =)\ 102 - 129$; that is x will be negative.

The above method will be better underſtood, if illuſtrated by a few examples.

QUESTION II.

Required the values of x, and y, in the equation $71\,x + 17\,y = 1005$? *See Queſt.* 229, *Vol.* 1.

SOLUTION.

Since $71\,x + 17\,y = 1005$,

Th. - $17\,y = 1005 - 71\,x$,

And by diviſion $y = \left(\frac{1005 - 71x}{17} =\right) 59 - 4x + \frac{2 - 3x}{17}$;

Put - $r = \frac{2 - 3\,x}{17}$; Then $3\,x = 2 - 17\,r$,

Whence $x = \left(\frac{2 - 17r}{3} =\right) - 5\,r + \frac{2 - 2\,r}{3}$;

Put - $s = \frac{2 - 2\,r}{3}$; Then $2\,r = 2 - 3\,s$,

And - $r = \left(\frac{2 - 3\,s}{2} =\right) 1 - s - \frac{s}{2}$; Th. let $p = \frac{s}{2}$;

Then - $s = 2\,p$;

$r = \left(\frac{2 - 3 \times 2\,p}{2} =\right) 1 - 3\,p$;

$x = 17\,p - 5$;

And - $y = \frac{1005 - 71 \times 17\,p + 355}{17}$,

Or - $y = \left(\frac{1360 - 71 \times 17\,p}{17} =\right) 80 - 71\,p$.

Now becauſe $x = 17\,p - 5$, Th. $p \sqsubset \frac{5}{17}$.

And becauſe $y = 80 - 71\,p$, Th. $p \sqsupset \left(\frac{80}{71} =\right) 1\frac{9}{71}$;

Therefore the only value of p in integers will be 1.

And then $x = 17 - 5 = 12$; And $y = 80 - 71 = 9$.

QUES-

QUESTION III.

How many ways can 100 pounds be paid by guineas at 21, and piſtoles at 17 ſhillings each? *See Queſt.* 225, *Vol.* 1.

SOLUTION.

By the queſt. $21x+17y=2000$,

Or - - $17y=2000-21x$;

Therefore by diviſion $\Big\}$ $y=\left(\frac{2000-21x}{17}=\right)117-x+\frac{11-4x}{17}$;

Let - $r=\frac{11-4x}{17}$; Then $4x=11-17r$,

Therefore $x=\left(\frac{11-17r}{4}=\right)2-4r+\frac{3-r}{4}$;

Let - $s=\frac{3-r}{4}$; Then $4s=3-r$,

Therefore $r=3-4s$:

Now - $x=\left(\frac{11-17\times\overline{3-4s}}{4}=\right)\frac{11-51+68s}{4}$,

Or - $x=\left(\frac{68s-40}{4}=\right)17s-10$;

Lastly - $y=\frac{2000-21\times 17s+210}{17}$,

Or - $y=\left(\frac{2210-21\times 17s}{17}=\right)130-21s$.

Now becauſe $x=17s-10$, Th. $s \sqsubset \frac{10}{17}$;

And becauſe $y=130-21s$, Th. $s \sqsupset \left(\frac{130}{21}=\right)6\frac{4}{21}$;

Therefore $s = 1,\ 2,\ 3,\ 4,\ 5,\ 6;$

And then $\begin{cases} x = 7,\ 24,\ 41,\ 58,\ 75,\ 92. \\ y = 109,\ 88,\ 67,\ 46,\ 25,\ 4. \end{cases}$

QUESTION IV.

How many values can x and y have, in affirmative integers, when the given equation is $5x + 8y = 1989$?

SOLUTION.

Since $5x + 8y = 1989$,

Th. - $5x = 1989 - 8y$,

And $\therefore$ $x = \left(\frac{1989 - 8y}{5} =\right) 397 - y + \frac{4-3y}{5}$;

Let $\therefore$ $r = \frac{4-3y}{5}$, Then $3y = 4 - 5r$,

Therefore $y = \left(\frac{4-5r}{3} =\right) 1 - r + \frac{1-2r}{3}$;

Let $\therefore$ $s = \frac{1-2r}{3}$, Then $2r = 1 - 3s$,

Therefore $r = \left(\frac{1-3s}{2} =\right) - s + \frac{1-s}{2}$;

Let $\therefore$ $t = \frac{1-s}{2}$, Then $2t = 1 - s$;

Therefore $s = 1 - 2t$;

And $\therefore$ $r = \left(\frac{1 - 3 \times \overline{1-2t}}{2} = \frac{1-3+6t}{2} =\right) 3t - 1$;

Also $\therefore$ $y = \left(\frac{4 - 15t + 5}{3} =\right) 3 - 5t$;

Lastly $x = \left(\frac{1989 - 24 + 40t}{5} =\right) 393 + 8t$.

Now ſince $y = 3 - 5t$; it follows, that t is either 0, or a negative integer; and x will become negative, when $8t$ exceeds 393; Therefore the values of t will be $\frac{1}{8} \times 393 + 1\ (= 49 + 1) = 50$ the number required.

Hence the firſt values are $y = 3$, and $x = 393$; and the laſt will be $y = (3 + 5 \times 49 =)\ 248$; and $x = (393 - 8 \times 49 =)\ 1$.

SCHOLIUM. There are two particular caſes, in which the ſolutions of theſe queſtions may be obtained, without the above proceſs.

Firſt, From the ſolution of queſt. 233, part 1, vol. 1. it will appear, That, if the abſolute number can be meaſured by the ſum of the coefficients of the two unknown quantities; then thoſe unknown quantities may be ſeverally equal to the number, by which the ſum of thoſe coefficients doth meaſure the ſaid abſolute number, and the other anſwers (if any) may be readily found therefrom.

For example, if $3z + 5y = (19 \times 8 =)\ 152$;

Then - $z = y = 19$;

$$\text{Th}\cdot \begin{cases} z = 4,\ 9,\ 14,\ 19,\ 24,\ 29,\ 34,\ 39,\ 44,\ 49. \\ y = 28,\ 25,\ 22,\ 19,\ 16,\ 13,\ 10,\ 7,\ 4,\ 1. \end{cases}$$

Secondly, From the ſolutions of queſtions 221 and 223, part 1, vol. 1, it will appear, that, if the abſolute number can be meaſured by either of the coefficients of the indetermined numbers; then the indetermined number, that has the meaſuring coefficient, may be equal to the difference between the number by which the coefficient meaſures the abſolute number, and the coefficient of the other indetermined number; and that other indetermined number will, at the ſame time, be equal to the meaſuring coefficient: whence the other anſwers are readily found.

EXAMPLE I. If $3x + 5y = 51$;

Then $5y = (51 - 3x =)\ \overline{17 - x} \times 3$:

Whence by queſt. 221. $y = 3,\ 6,\ 9,\ 12,$ &c.

And $17 - x = 5,\ 10,\ 15,\ 20.$ &c.

Th· $(17 - 5 =)\ x = 12,\ 7,\ 2.$

EXAMPLE 2. If $3x + 5y = 85$.
Then $3x = 85 - 5y = \overline{17 - y} \times 5$;
Whence by queſt. 221. $x = 5,\ 10,\ 15,\ 20,$ &c.
And $17 - y = 3,\ 6,\ 9,\ 12,$ &c.
Th. $y = 14,\ 11,\ 8,\ 5,$ &c.

COROL. I.

Hence, if both the coefficients meaſure the abſolute number, then the leaſt value of each indetermined quantity will be the coefficient of the other.

EXAMPLE, If $3x + 4y = 120$;
Then the leaſt value of x will be 4, and the leaſt value of y will be 3;
And $x = 4,\ 8,\ 12,\ 16,\ 20,\ 24,\ 28,\ 32,\ 36.$
$y = 27,\ 24,\ 21,\ 18,\ 15,\ 12,\ 9,\ 6,\ 3$;

COROL. II.

If both the coefficients, and alſo their ſum, will meaſure the abſolute number; then (beſides the property exhibited in the firſt corollary) the number of anſwers will be equal to the ſum of the two coefficients, leſs one.

EXAMPLE I. If $3x + 5y = 120$;
Then, the leaſt value of x will be 5; the leaſt value of y will be 3; and the number of anſwers will be $(3 + 5 - 1 =)$ 7.
For $x = 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 35.$
And $y = 21,\ 18,\ 15,\ 12,\ 9,\ 6,\ 3.$

EXAMPLE II. If $4x + 7y = 308$;
Then $x = 7,\ 14,\ 21,\ 28,\ 35,\ 42,\ 49,\ 56,\ 63,\ 70.$
And $y = 40,\ 36,\ 32,\ 28,\ 24,\ 20,\ 16,\ 12,\ 8,\ 4.$
Where the number of anſwers is $(4 + 7 - 1 =)$ 10.

But even where it will be convenient to uſe the firſt exhibited proceſs, ſome caſes will be much eaſier than others.

For inſtance, if the greater coefficient, and the abſolute number (being both divided by the leſſer coefficient) leave the ſame remainder, (as in the following queſtion) the proceſs will be much ſhorter than in ſome of the former caſes.

QUESTION V.

How many different ſolutions, in affirmative integers, can be given to the equation $3x+5y=173$?

SOLUTION.

Here $x=\left(\frac{173-5y}{3}=\right)57-y+\frac{2-2y}{3}$:

Let $r=\frac{2-2y}{3}$; Then $3r=2-2y$;

Th. $y=\left(\frac{2-3r}{2}=\right)1-r-\frac{r}{2}$; If $\frac{r}{2}=s$,

$r=2s$;

$y=\left(\frac{2-6s}{2}=\right)1-3s$;

And $x=\left(\frac{173-5\times\overline{1-3s}}{3}=\right)56+5s$:

Where s muſt be 0, or a negative number; Therefore the ſigns being changed,

$y=1+3s$; Th. $s \sqsubset -\frac{1}{3}$:

$x=56-5s$; $s \sqsupset \left(\frac{56}{5}=\right)11\frac{1}{5}$:

And the number of anſwers will be $1+11=12$.

QUESTION VI.

It is required to find three affirmative integers, ſuch, that the firſt being multiplied by 3, the ſecond by 5, and the third by 8, the ſum of all the products may be 10003?

SOLUTION.

By the queſt. $3x+5y+8z=10003$,

Or $3x=10003-5y-8z$,

Th. (by diviſion) $x=3334-y-2z+\frac{1-2y-2z}{3}$;

Let $r=\frac{1-2y-2z}{3}$; Then $2y=1-2z-3r$,

Whence $y=\left(\frac{1-2z-3r}{2}=\right)-z-r+\frac{1-r}{2}$;

Let $s=\frac{1-r}{2}$, Then $2s=1-r$,

Therefore $r=1-2s$:

And $y=\left(\frac{1-2z-3\times\overline{1-2s}}{2}=\right)\frac{1-2z-3+6s}{2}$,

That is $y=\left(\frac{6s-2z-2}{2}=\right)3s-z-1$;

And $x=\frac{10003-8z-5\times\overline{3s-z-1}}{3}$,

Or $x=\frac{10008-3z-15s}{3}$,

That is $x=3336-z-5s$.

In which two equations (*viz.* $y=3s-z-1$, and $x=3336-z-5s$) s is indetermined, and z is to be determined with proper limitation.

The leaſt interpretation that z can poſſibly have being 1; the above equations may, by that aſſumption, ſhew the limits of s; for then $3s - z - 1$ will become $3s - 2$, and $s \sqsubset \frac{2}{3}$; And $3336 - z - 5s$ will become $3335 - 5s$, Th. $s \sqsupset \left(\frac{3335}{5} =\right) 667$.

Having thus found the limits of s; thoſe of z will appear as follow:

A B

If $s = 1$; $x = 3331 - z$: And $y = 2 - z$.
$s = 2$; $x = 3326 - z$: $y = 5 - z$.
$s = 3$; $x = 3321 - z$: $y = 8 - z$.
$s = 4$; $x = 3316 - z$: $y = 11 - z$.
&c. &c. &c.

For the clearer explanation of what follows, let the values of x, collectively taken, be denominated by the letter A, which is placed over them; and the values of y, by the letter B, which is placed over them; Then,

Firſt, Since the abſolute numbers in the column A decreaſe perpetually by the ſubtraction of 5; the number of terms in that column (after the firſt) may be found by dividing 3331 (the greateſt abſolute number) by 5 their common difference: Now the quotient of that diviſion will be 666, and the remainder 1; therefore there will be 666 values of x, in that column, after the firſt, or 667 in the whole.

Secondly, Since the remainder left after dividing 3331, the greateſt abſolute number in the column A, by 5, the common difference of thoſe numbers, is 1; it appears, that the laſt value of x in that column will be $1 - z$; which value of x is uſeleſs, for then x will be either 0, or negative; and therefore there will be but 666 uſeful values of x in that column; and s may be any number leſs than 667: The three laſt values of s, x, and y, follow;

$$\begin{array}{llll} \text{If } s = 664; & x = 16 - z: & \text{And } y = 1991 - z. \\ s = 665; & x = 11 - z: & y = 1994 - z. \\ s = 666; & x = 6 - z: & y = 1997 - z. \end{array}$$

Thirdly, If we were to determine the limits of z from the firſt equation of the column A, *viz* $x = 3331 - z$, we might be apt to conclude, that z might be any number under 3331; and by conſequence that it might receive 3330 different interpretations; but conſidering the correſponding equation in the column B, *viz.* $y = 2 - z$: it appears that the values of z are thereby reſtrained, and that it muſt be leſs than 2: In this caſe therefore z admits but of one interpretation, *viz.* 1, whence $x = (3331 - 1 =)\ 3330$; and $y = (2 - 1 =)\ 1$;

$$\begin{array}{rlr} \text{Now } 3x = & (3 \times 3330 =) & 9990, \\ 5y = & (5 \times 1 =) & 5, \\ \text{And } 8z = & (8 \times 1 =) & 8. \\ \hline & & 10003. \end{array}$$

From which operation, we may conclude, that the above values of x, y, and z, are rightly found.

Fourthly, By examining the ſecond values of x and y, (which are $x = 3326 - z$, and $y = 5 - z$) it will appear, that z may be any number under 5, *viz.* either 1, 2, 3, or 4: Again, taking the third value of y, (*viz.* $y = 8 - z$) it is plain, that z may be any number leſs than 8, and conſequently, that it will have 7 values: And hence we may be apt to conclude that the number of interpretations of which z is capable, when s is ſeverally interpreted by 1, 2, 3, 4, 5, &c. will be 1, 4, 7, 10, 13, &c. But,

Fifthly, In the ſame manner as the values of z, in the column A, are reſtrained by thoſe in the column B, in the caſes before cited; ſo, after a certain number of terms, will its values, in the column B, be reſtrained by thoſe in the column A: For, if we were to determine the limits of z from the laſt value of x and y, (*viz.* that wherein $s = 666$) we ſhall find, in column B, that $y = 1997 - z$; from whence it might be concluded, that z may be any number leſs than 1997, if we did not, in the column A, find, that $x = 6 - z$, and conſequently, that z muſt be leſs than 6.

Sixthly, From the above examples, it is evident, that the column A will begin to reſtrain the values of z, in the column B, as ſoon as the abſolute number, in the column A, becomes leſs than the abſolute number in the column B; it remains, therefore, to find when that will happen; in order to which,

Let $n =$ the diſtance of the terms, from the firſt, wherein the abſolute numbers of the two columns will become equal.

Then, becauſe the abſolute numbers of the column A, are a decreaſing arithmetical progreſſion, whoſe greateſt term is 3331, and common difference 5; the term, whoſe diſtance from the greateſt is n, will be $3331 - 5n$.

And becauſe the abſolute numbers of the column B are an encreaſing arithmetical progreſſion, whoſe leaſt term is 2, and common difference 3; the term, whoſe diſtance from the leaſt is n, and will be $2 + 3n$.

Hence $3331 - 5n = 2 + 3n$,

Or $3331 - 2 = (5n + 3n =) 8n$,

That is $\left(\frac{3329}{8} = \right) 416\frac{1}{8} = n$.

Therefore, for 416 terms, after the firſt, (that is to ſay, for 417 terms) the abſolute numbers in the column A, exceed thoſe in the column B; but in the 418th term, and all following, the abſolute numbers in the column B, exceed thoſe in the column A; for the more clear perception of which, the 416th, 417th, and ſome following terms, are here ſet down.

$n = 416$; $x = 1256 - z$; $y = 1247 - z$

$n = 417$; $x = 1251 - z$; $y = 1250 - z$

$n = 418$; $x = 1246 - z$; $y = 1253 - z$

$n = 419$; $x = 1241 - z$; $y = 1256 - z$

Hence, for the firſt 417 terms, the column B reſtrains the values of z in the column A; and for the laſt (666 — 417 =) 249 terms the column A reſtrains the values of z in the column B.

Laſtly, Since for the firſt 417 terms, wherein the number of the different values of z are determined by the column B, thoſe numbers form an arithmetical progreſſion, *viz.* 1, 4, 7, 10, 13, &c. the laſt term of which is (1250 — 1 =) 1249; Therefore, the ſum of that progreſſion will expreſs the number of all the values that z can poſſibly have in thoſe 417 terms; which ſum is per queſt. 12. part 2. vol. 1. $\frac{417 \times \overline{1249 + 1}}{2}$.

And ſince, for the laſt 249 terms, wherein the number of the different values of z are determined by the column A, they form (if we begin at the leaſt) the arithmetical progreſſion (6 — 1; 11 — 1, &c.) 5· 10· 15· 20· &c. of which the laſt is (1246 — 1 =) 1245; Therefore the number of all the values of z, therein, will be $\frac{249 \times \overline{1245 + 5}}{2}$.

Therefore, the number of different anſwers that may be given to the queſtion propounded, will be $\frac{417 \times \overline{1249 + 1}}{2} + \frac{249 \times \overline{1245 + 5}}{2} = \frac{\overline{417 + 249} \times 1250}{2} = 416250$.

QUESTION VII.

How many different values of x, y, and z, in affirmative integers, can ariſe from the equation $3x + 5y + 19z = 13051$?

SOLU-

SOLUTION.

Here $3x = 13051 - 5y - 19z$,

And $x = 4350 - y - 6z + \frac{1 - 2y - z}{3}$;

Let $r = \frac{1 - 2y - z}{3}$; Then $2y = 1 - z - 3r$,

Th. $y = \left(\frac{1 - z - 3r}{2} =\right) - r + \frac{1 - z - r}{2}$;

Let $s = \frac{1 - z - r}{2}$; Then $2s = 1 - z - r$,

Th. $r = 1 - z - 2s$;

And $y = \left(\frac{1 - z - 3 \times \overline{1 - z - 2s}}{2} =\right) \frac{1 - z - 3 + 3z + 6s}{2}$,

Or $y = \left(\frac{6s + 2z - 2}{2} =\right) 3s + z - 1$;

Lastly $x = \frac{13051 - 15s - 5z + 5 - 19z}{3}$,

Or $x = \left(\frac{13056 - 15s - 24z}{3} =\right) 4352 - 5s - 8z$.

Now, becauſe z has different ſigns in the values of x and y; let us ſubſtitute $s = p - z$, which is done as a matter of convenience, not of neceſſity.

Then $x = (4352 - 5 \times \overline{p - z} - 8z =)\ 4352 - 5p - 3z$,
And $y = (3 \times \overline{p - z} + z - 1 =)\ 3p - 2z - 1$;
Whence $p \sqsubset (\frac{3}{3} =)\ 1$; And $p \sqsupset 869\frac{4}{5}$.

A B

Now if $p = 2$; $x = 4342 - 3z$; And $y = 5 - 2z$.
$p = 3$; $x = 4337 - 3z$; $y = 8 - 2z$.
$p = 4$; $x = 4332 - 3z$; $y = 11 - 2z$.
&c. *&c.* *&c.*

If $p = 867$; $x = 17 - 3z$; $y = 2600 - 2z$.
$p = 868$; $x = 12 - 3z$; $y = 2603 - 2z$.
$p = 869$; $x = 7 - 3z$; $y = 2606 - 2z$.
Where the number of terms is 868.

Now from the Beginning of the column B, we find that

When $p = 2$; Then $z \sqsupset (\frac{5}{2} =) 2\frac{1}{2}$. Th. z has 2
$p = 3$; $z \sqsupset (\frac{8}{2} =) 4$. z 3
$p = 4$; $z \sqsupset (\frac{11}{2} =) 5\frac{1}{2}$. z 5
$p = 5$; $z \sqsupset (\frac{14}{2} =) 7$. z 6
$p = 6$; $z \sqsupset (\frac{17}{2} =) 8\frac{1}{2}$. z 8
&c. *&c.* *&c.*
Values.

And from the end of column A, we find that when

$p = 869$; Then $z \sqsupset (\frac{7}{3} =) 2\frac{1}{4}$. Th. z has 2
$p = 868$; $z \sqsupset (\frac{12}{3} =) 4$. z 3
$p = 867$; $z \sqsupset (\frac{17}{3} =) 5\frac{2}{3}$. z 5
$p = 866$; $z \sqsupset (\frac{22}{3} =) 7\frac{1}{3}$. z 7
$p = 865$; $z \sqsupset (\frac{27}{3} =) 9$. z 8
$p = 864$; $z \sqsupset (\frac{32}{3} =) 10\frac{2}{3}$. z 10
$p = 863$; $z \sqsupset (\frac{37}{3} =) 12\frac{1}{3}$. z 12
$p = 862$; $z \sqsupset (\frac{42}{3} =) 14$. z 13
&c. *&c.* *&c.*
Values.

But although the two ſeries 2, 3, 5, 6, 8, *&c.* and 2, 3, 5, 7, 8, 10, 12, 13, *&c.* which contain the number of the different values of z, ariſing from each term of the ſaid columns, are not themſelves arithmetical progreſſions as thoſe reſulting from the laſt queſtion were; yet they may be divided into ſeries that are ſo: That is to ſay the ſeries 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, *&c.* may be divided into the two which follow, *viz.* 2, 5, 8, 11, 14, *&c.* and 3, 6, 9, 12, 15, *&c.* both which are arithmetical; And the ſeries 2, 3, 5, 7, 8, 10, 12, 13, 15, 17, 18, 20, *&c.* may be divided into the following three ſeries which are arithmetical, *viz.* 2, 7, 12, 17, *&c.* 3, 8, 13, 18, *&c.* and 5, 10, 15, 20, *&c.* And therefore may be ſummed, as the former were, when the greateſt term of each ſeries is known.

Now, becauſe the values of z, in the column A, are determined by $\frac{1}{3}$ of the abſolute number, and in the column B by $\frac{1}{2}$ thereof; it will follow, that in order to find where the column B ceaſes to determine the number of the values of z, and the column A begins ſo to do, it will be proper to make an equation between $\frac{1}{3}$ of the abſolute number belonging to the term whoſe diſtance from the greateſt in the column A is n, and $\frac{1}{2}$ of the abſolute number of the correſponding term in the column B; that is, becauſe the abſolute numbers differ by 5 and 3,

$$\frac{4342-5n}{3} = \frac{5+3n}{2},$$

Or $8684 - 10n = 15 + 9n$;

That is $8684 - 15 = (10n + 9n =)\ 19n$;

Therefore $\frac{8669}{19} = 456\frac{5}{19} = n$.

Whence it follows, that the column B will continue to determine the number of the values of z for 456 terms after the firſt, that is, for 457 terms; and that the column A will determine the ſame, for the remaining (868 — 457 =) 411 terms.

Now, ſince the two ſeries, depending on the column B, are between them to contain 457 terms, it follows that the firſt of them will contain $\left(\frac{457}{2} + 1 =\right)$ 229 terms, and the ſecond 228. Whence the greateſt term of the former will be ($2+228\times3=$) 686, and the greateſt term of the latter ($3+227\times3=$) 684.

Th· $\frac{\overline{686+2}\times229}{2}+\frac{\overline{684+3}\times228}{2}=157094$ will be the Sum of both ſeries.

Again, ſince the three ſeries, which depend on the column A, are between them to contain 411 terms; it follows, that each of them muſt contain $\left(\frac{411}{3}=\right)$ 137 terms; whence the greateſt terms of them will be ($2+5\times136=$) 682; ($3+5\times136=$) 683; And ($5+5\times136=$) 685; whence their ſum will be $\overline{\overline{682+2}+\overline{683+3}+\overline{685+5}}\times\frac{137}{2}=141110$: and $157094+141110=298204$ will be the number of different anſwers, which the propoſed queſtion will admit of.

Thus far Mr. De Moivre.

As the above quoted juſtly celebrated Mathematician has aſſerted, that the ſubſtitution $s=p-z$ (whereby the ſign — becomes prefixed to the indetermined quantity z, in the values of both x and y) is not made for neceſſity, but by choice; it may be a ſpeculation worth purſuing, to find the conſequences of omitting the ſame.

The queſtion was reduced to the two following equations,

$y=3s+z-1$. Where calling z one $s\sqsubset0$;

$x=4352-5s-8z$, And $s\sqsupset\left(\frac{4344}{5}=\right)868\frac{4}{5}$.

If $s = 1$; $x = 4347 - 8z$; $y = 2 + z$; And z has 543 values.
$s = 2$; $x = 4342 - 8z$; $y = 5 + z$; 542
$s = 3$; $x = 4337 - 8z$; $y = 8 + z$; 542
$s = 4$; $x = 4332 - 8z$; $y = 11 + z$; 541
$s = 5$; $x = 4327 - 8z$; $y = 14 + z$; 540
$s = 6$; $x = 4322 - 8z$; $y = 17 + z$; 540
$s = 7$; $x = 4317 - 8z$; $y = 20 + z$; 539
$s = 8$; $x = 4312 - 8z$; $y = 23 + z$; 538
$s = 9$; $x = 4307 - 8z$; $y = 26 + z$; 538
$s = 10$; $x = 4302 - 8z$; $y = 29 + z$; 537
$s = 11$; $x = 4297 - 8z$; $y = 32 + z$; 537
$s = 12$; $x = 4292 - 8z$; $y = 35 + z$; 536
$s = 13$; $x = 4287 - 8z$; $y = 38 + z$; 535
$s = 14$; $x = 4282 - 8z$; $y = 41 + z$; 535
$s = 15$; $x = 4277 - 8z$; $y = 44 + z$; 534
$s = 16$; $x = 4272 - 8z$; $y = 47 + z$; 533
$s = 17$; $x = 4267 - 8z$; $y = 50 + z$; 533
&c. *&c.* *&c.* *&c.*

Now the above ſeries of the values of z contains 8 arithmetical progreſſions, *&c.*

543, 538, 533, *&c.* 542, 537, 532, *&c.* 542, 537, 532, *&c.* 541, 536, 531, *&c.* 540, 535, 530, *&c.* 540, 535, 530, *&c.* 539, 534, 529, *&c.* And 538, 533, 528, *&c.*

Of which (ſince all the ſeries contains 868 terms) the four laſt progreſſions will contain $\left(\frac{868}{8} =\right)$ 108 terms, and the four firſt 109 terms each.

Alſo (among the four firſt) the progreſſion 542, 537, 532, *&c.* and (among the four laſt) 540, 535, 530, *&c.* are repeated twice; conſequently the double of each term, *&c.* 1084, 1074, 1064, *&c.* and 1080, 1070, 1060, *&c.* may be wrote in their ſtead.

Again, ſince the firſt and laſt progreſſion, differ only in the firſt term, we may write 543 more 1076, 1066, 1056, &c. for them.

The ſame method may be advantageouſly purſued, with reſpect to the two remaining progreſſions, that have 109 terms; *viz.* if their firſt terms be ſeparately conſidered, then all the progreſſions will have the ſame number of terms, *viz.* 108.

Hence the number of anſwers required may be expreſſed as follows,

$$\begin{array}{rl} 543 + 108 \text{ terms of} & 1076,\ 1066,\ 1056,\ \&c. \\ +\ 1084 + 108 \text{ terms of} & 1074,\ 1064,\ 1054,\ \&c. \\ +\ 541 + 168 \text{ terms of} & 536,\ 531,\ 526,\ \&c. \\ + 108 \text{ terms of} & \left\}\begin{array}{l} 1080,\ 1070,\ 1060,\ \&c. \\ \\ 539,\ 534,\ 529,\ \&c. \end{array}\right. \end{array}$$

Which may (by addition) be expreſſed by the following ſingle ſeries:

2168 + 108 terms of 4305, 4265, 4225, &c. of which ſince the common difference is 40, the laſt term will be $(4305 - (107 \times 40 =) 4280 =)\ 25$; And conſequently the ſum of the ſeries will be $\frac{\overline{4305 + 25} \times 108}{2} = 233820$; to which, if the above reſerved ſum (2168) be added, the reſult will be 235988.

But this differs from 298204, the number of anſwers before found, by 62216; from whence, it may very juſtly be concluded, that this progreſs is defective.

Now, in the equation $y = 3s + z - 1$, when z is any number greater than 1, s may be nothing, or a negative number; And how great that negative number may be, depends upon the magnitnde of z; for, by the equation $y = 3s + z - 1$; $s \sqsubset -\frac{z-1}{3}$.

Now the greater limit of z may be obtained from the original equation, $3x + 5y + 19z = 13051$; For $19z = 13051 - 3x - 5y$, and ſince the leaſt value of either

x or

z or y is unity, the greateſt value of $19z$ will be $(13051-3-5=)\ 13043$; And therefore z cannot exceed $\left(\frac{13043}{19}=\right)686\frac{9}{19}$.

Hence if we write 686 for z, in the expreſſion, $s \sqsubset -\frac{z-1}{3}$; it will be $s \sqsubset \left(-\frac{685}{3}=\right)-228\frac{1}{3}$.

Now the equation, $y=3s+z-1$, may, when s is 0 or negative, become $y=z-3s-1$; And the equation, $x=4352-5s-8z$, may become $x=4352+5s-8x$. Then

If $s=0$; $y=z-1$; $x=4352-8z$; $z \sqsubset 1$ & $\sqsupset 544$; has values 542.
$s=1$; $y=z-4$; $x=4357-8z$; $z \sqsubset 4$ & $\sqsupset 544\frac{5}{8}$; 540.
$s=2$; $y=z-7$; $x=4362-8z$; $z \sqsubset 7$ & $\sqsupset 545\frac{2}{8}$; 538.
$s=3$; $y=z-10$; $x=4367-8z$; $z \sqsubset 10$ & $\sqsupset 545\frac{7}{8}$; 535.
$s=4$; $y=z-13$; $x=4372-8z$; $z \sqsubset 13$ & $\sqsupset 546\frac{4}{8}$; 533.
$s=5$; $y=z-16$; $x=4377-8z$; $z \sqsubset 16$ & $\sqsupset 547\frac{1}{8}$; 531.
$s=6$; $y=z-19$; $x=4382-8z$; $z \sqsubset 19$ & $\sqsupset 547\frac{6}{8}$; 528.
$s=7$; $y=z-22$; $x=4387-8z$; $z \sqsubset 22$ & $\sqsupset 548\frac{3}{8}$; 526.
$s=8$; $y=z-25$; $x=4392-8z$; $z \sqsubset 25$ & $\sqsupset 549$; 523.
$s=9$; $y=z-28$; $x=4397-8z$; $z \sqsubset 28$ & $\sqsupset 549\frac{2}{8}$; 521.
$s=10$; $y=z-31$; $x=4402-8z$; $z \sqsubset 31$ & $\sqsupset 550\frac{2}{8}$; 519.
$s=11$; $y=z-34$; $x=4407-8z$; $x \sqsubset 34$ & $\sqsupset 550\frac{7}{8}$; 516.
$s=12$; $y=z-37$; $x=4412-8z$; $z \sqsubset 37$ & $\sqsupset 551\frac{4}{8}$; 514.
$s=13$; $y=z-40$; $x=4417-8z$; $z \sqsubset 40$ & $\sqsupset 552\frac{1}{8}$; 512.
$s=14$; $y=z-43$; $x=4422-8z$; $z \sqsubset 43$ & $\sqsupset 552\frac{6}{8}$; 509.
$s=15$; $y=z-46$; $x=4427-8z$; $z \sqsubset 46$ & $\sqsupset 553\frac{3}{8}$; 507.
$s=16$; $y=z-49$; $x=4432-8z$; $z \sqsubset 49$ & $\sqsupset 554$; 504.
&c. &c. &c. &c. &c. &c.
$s=228$; $y=z-685$; $x=5492-8z$; $x \sqsubset 685$ $\sqsupset 686\frac{4}{8}$; has 1 value.

Whence it is evident, that the number of anſwers reſulting from the conſideration that z may be 0, or negative, conſiſt of 8 arithmetical progreſſions, whoſe greateſt terms are as above, their common difference is 19, and the number of the terms of all the 8 progreſſions, taken together 229; but ſince $\frac{229}{8} = 28\frac{5}{8}$, therefore the firſt five progreſſions will have 29 terms, and the remaining three 28 terms each.

Now if 2688, the ſum of the greateſt terms of the 5 firſt progreſſions, be ſeparately reſerved, there will remain eight progreſſions, each of 28 numbers; the greateſt terms of which, *viz.* 523, 521, 519, 516, 514, 531, 528, 526, being added together, make 4178, the greateſt terms of a progreſſion, which will be equal to them all; and becauſe each of the 8 former progreſſions has 19 for its common difference, therefore $(19 \times 8 =)\ 152$ will be the common difference of the latter.

Now $(4178 - (27 \times 152 =)\ 4104 =)\ 74$, will be the leaſt term, and conſequently, $\frac{\overline{4178+74} \times 28}{2} = 59528$ will be the ſum: that is, the ſum of the 8 above-mentioned progreſſions, whoſe common difference and number of terms are equal; to which, if 2688 (the ſum of the greateſt terms of thoſe five progreſſions, that have 29 terms each) be added, their ſum will be 62216, which was before obſerved to be the deficiency of the former proceſs.

As the introduction of the latter method, after that which the above celebrated gentleman has given, may, by many, be thought impertinent, the following queſtion, wherein a proper ſubſtitution (ſuch as is uſed above to make z negative in the values of both x and y) doth not readily appear, and perhaps cannot be found at all, is given by way of juſtification.

QUESTION VIII.

How many different ſolutions in affirmative integers, will the equation, $5x + 7y + 9z = 93256$, admit of?

SOLUTION.

Here $x = 18651 - y - 2z + \frac{1 - 2y - 4z}{5}$,

Let $r = \frac{1 - 2y - 4z}{5}$; Then $2y = 1 - 4z - 5r$;

Whence $y = \left(\frac{1 - 4z - 5r}{2} =\right) - 2z - 2r + \frac{1 - r}{2}$,

Let $s = \frac{1 - r}{2}$; Then $2s = 1 - r$;

Therefore $r = 1 - 2s$;

$$y = \left(\frac{1 - 4z - 5 \times \overline{1 - 2s}}{2} =\right) 5s - 2z - 2;$$

And $x = \frac{93256 - 7 \times \overline{5s - 2z - 2} - 9z}{5}$,

Or $x = 18654 - 7s + z$.

Which (the following ſubſtitutions being made) will become

$s = p - z$; $y = 5p - 7z - 2$; And $x = 18654 - 7p + 8z$;
$s = p + z$; $y = 5p + 3z - 2$; And $x = 18654 - 7p - 6z$;
$u = s - z$; $y = 3s + 2u - 2$; And $x = 18654 - 6s - u$;
$w = s = z$; $y = 7s - 2w - 2$; And $x = 18654 - 8s + w$;
$s = z - p$; $y = 3z - 5t - 2$; And $x = 18654 + 7t - 6z$;

In all which, and many more that have been tried, the unknown quantity has different ſigns in the values of x and y.

It will be expedient, therefore, as in the laſt proceſs, to find the greater limit of z; which (ſince $9z$ may be $= (93256 - 7 - 5 =)\ 93244$; And $\frac{93244}{9} = 10360\frac{4}{9}$) will be 10360: Whence the equation, $x = 18654 - 7s + z$, may become, $x = (18654 - 7s + 10360 =)\ 29014 - 7s$; And when $z = 1$, the leaſt value poſſible of, $y = 5s - 2z - 2$, will be $y = 5s - 4$.

Now by the equations $\left\{\begin{array}{l} y = 5s - 4;\ s \sqsubset \frac{4}{5}. \\ x = 29014 - 7s,\ s \sqsupset \left(\frac{29014}{7} =\right) 4144\frac{6}{7} \end{array}\right.$

If values

$s=1; y= 3-2z; x=18647+z; z \sqsupset 1\frac{1}{2}, \& \sqsubset 0$, has 1

$s=2; y= 8-2z; x=18640+z; z \sqsupset 4, \& \sqsubset 0$, has 3

$s=3; y=13-2z; x=18633+z; z \sqsupset 6\frac{1}{2}, \& \sqsubset 0$, has 6

$s=4; y=18-2z; x=18626+z; z \sqsupset 9, \& \sqsubset 0$, has 8

$s=5; y=23-2z; x=18619+z; z \sqsupset 11\frac{1}{2}, \& \sqsubset 0$, has 1

&c. &c. &c. &c. &c.

Therefore (ſo long as the abſolute number in the value of x continues affirmative) the number of anſwers, which reſult from the different aſſumptions of s, will form the two following arithmetical progreſſions, *viz.* 1, 6, 11, 16, &c. And 3, 8, 13, &c.

Now 18654, the abſolute number in the equation, exhibiting the value of x, being divided by ſeven, will quote $2664\frac{6}{7}$; And therefore the number 7 can be taken from it but 2664 times, and leave an affirmative remainder; whence in the 2665th value of s it will be negative.

If s is	Then	Values.
2665;	$y=13323-2z$; $x=z-1$; $z \sqsupset 6661\frac{1}{2}$, & $\sqsubset$ 1,	6660.
2666;	$y=13328-2z$; $x=z-8$; $z \sqsupset 6664$, & $\sqsubset$ 8,	6655.
2667;	$y=13333-2z$; $x=z-15$; $z \sqsupset 6666\frac{1}{2}$, & $\sqsubset$ 15,	6651.
2668;	$y=13338-2z$; $x=z-22$; $z \sqsupset 6569$, & $\sqsubset$ 22,	6646.
2669;	$y=13343-2z$; $x=z-29$; $z \sqsupset 6671\frac{1}{2}$, & $\sqsubset$ 29,	6642.
&c.	&c. &c. &c. &c.	&c.

Which Series is to be continued to $(4144 - 2664 =)$ 1480 Terms; of which the laſt follows,

$s = 4144$; $y = 20718 - 2z$; $x = z - 10354$; $z \sqsupset 10359$, $\sqsubset 10354$, & has 4 values.

Hence when the reſulting progreſſions in each caſe are combined, we may conclude that 1332 terms of the progreſſion 4, 14, 24, &c. and 740 terms of the progreſſion 13315, 13297, 13279, &c. will be the number of anſwers required $= 8869788 + 4931360 = 13801148$.

The following method of ſolving this queſtion will prove the above, and will ſerve as an introduction to what follows, concerning 4 indetermined quantities.

SOLUTION II.

In the equation $5x + 7y + 9z = 93256$,

When z is 1; Then $5x + 7y = (93256 - 9)\ 93247$

Th. $$x = \left(\frac{93247 - 7y}{5} =\right) 18649 - y + \frac{2-2y}{5}:$$

Let $$r = \frac{2-2y}{5};\ \text{Then } 2y = 2 - 5r,$$

And $$y = \left(\frac{2-5r}{2} =\right) 1 - 2r - \frac{r}{2}:$$

Let $$s = \frac{r}{2};\ \text{Then } 2s = r,\ \text{And}$$

Since $r = 2s$; $$y = \left(\frac{2-10s}{2} =\right) 1 - 5s,$$

And $x = \left(\frac{93247 - 7 \times \overline{1 - 5s}}{5} = \right) 18648 + 7s$

Where $s \sqsupset \frac{1}{5}$, and $s \sqsubset - \left(\frac{18648}{7} = \right) 2664$;

Hence when $s = 0$; $y = 1$; $x = 18648$.

$s = -1$; $y = 6$; $x = 18641$.

&c. &c.

$-s = -2662$; $y = 13311$; $x = 14$.

$s = -2663$; $y = 13316$; $x = 7$.

And, in this caſe, the queſtion has 2664 anſwers.

When $z = 2$; Then $5x + 7y = (93256 - 18 =) 93238$.

Th. $x = \left(\frac{93238 - 7y}{5} = \right) 18647 - y + \frac{3 - 2y}{5}$;

Let $r = \frac{3 - 2y}{5}$; Then $2y = 3 - 5r$,

And $y = \left(\frac{3 - 5r}{2} = \right) 1 - 2r + \frac{1 - r}{2}$

Let $s = \frac{1 - r}{2}$; Then $2s = 1 - r$,

And $r = 1 - 2s$:

Whence $y = \left(\frac{3 - 5 \times \overline{1 - 2s}}{2} = \right) 5s - 1$;

And $x = \left(\frac{93238 - 7 \times \overline{5s - 1}}{5} = \right) 18649 - 7s$;

Where $s \sqsubset \frac{1}{5}$; And $s \sqsupset \left(\frac{18649}{7} = 2664\frac{1}{7}\right)$;

Hence when $s = 1$; $y = 5 - 1 = 4$; & $x = 18649 - 7 = 18642$.

And when $s = 2664$; $y = 13320 - 1 = 13319$; & $x = 1$.

Th. when $z = 2$ the queſtion will have 2664 anſwers.

When $z = 3$; Then $5x + 7y = (93256 - 27 =) 93229$.

Therefore $x = \left(\frac{93229 - 7y}{5} = \right) 18645 - y + \frac{4 - 2y}{5}$:

Let $r = \frac{4 - 2y}{5}$; Then $2y = 4 - 5r$;

And $y = \left(\frac{4 - 5r}{2} =\right) 2 - 2r - \frac{r}{2}$:

Let $s = \frac{r}{2}$; Then $r = 2s$;

$y = \left(\frac{4 - 10s}{2} =\right) 2 - 5s$;

And $x = \left(\frac{93229 - 7 \times \overline{2 - 5s}}{5} =\right) 18643 + 7s$.

Where $s \sqsupset \frac{2}{5}$; And $s \sqsubset -2663\frac{2}{7}$.

Hence when $s = 0$; $y = 2$; And $x = 18643$;
$s = -2663$; $y = (2 + 13315 =) 13317$; $x = 2$.

Th. when $z = 3$, the queſtion admits of 2664 anſwers.

When $z = 4$; Then $5x + 7y = (93256 - 36 =) 93220$.

Therefore $x = \left(\frac{93220 - 7y}{5} =\right) 18644 - y - \frac{2y}{5}$;

Let $r = \frac{2y}{5}$; Then $5r = 2y$,

And $y = \left(\frac{5r}{2} =\right) 2r + \frac{r}{2}$;

Let $s = \frac{r}{2}$; Then $r = 2s$;

$y = \left(\frac{10s}{2} =\right) 5s$;

And $x = \left(\frac{93220 - 7 \times 5s}{5} =\right) 18644 - 7s$;

Where $s \sqsubset 0$; And $s \sqsupset \left(\frac{18644}{7} =\right) 2663\frac{3}{7}$.

Hence when $s = 1$; $y = 5$; $x = (18644 - 7 =) 18637$;
$s = 2663$; $y = (2663 \times 5 =) 13315$; $x = 3$.

Th. when $z = 4$, the queſtion admits of 2663 anſwers.

When $z=5$; Then $5x+7y=(93256-45=)93211$.

Therefore $x=\left(\frac{93211-7y}{5}=\right)18642-y+\frac{1-2y}{5}$;

Let $r=\frac{1-2y}{5}$; Then $2y=1-5r$,

And $y=\left(\frac{1-5r}{2}=\right)-2r+\frac{1-r}{2}$:

Let $s=\frac{1-r}{2}$; Then $2s=1-r$,

And $r=1-2s$;

$y=\left(\frac{1-5\times\overline{1-2s}}{2}=\right)5s-2$;

And $x=\left(\frac{93211-7\times\overline{5s-2}}{5}=\right)18645-7s$:

Where $s\sqsubset\frac{2}{5}$; and $s\sqsupset\left(\frac{18645}{7}=\right)2663\frac{4}{7}$.

Hence when $s=1$; $y=(5-2=)\ 3$; $x=18638$;

$s=2663$; $y=(2663\times5-2=)13313$; $x=4$.

Th· when $z=5$, the queſtion admits of 2663 anſwers.

When $z=6$; Then $5x+7y=(93256-54=)\ 93202$.

Therefore $x=\left(\frac{93202-7y}{5}=\right)18640-y+\frac{2-2y}{5}$;

Whence (proceeding as when z was equal to 1)

$y=1-5s$;

And $x=\frac{93202-7\times\overline{1-5s}}{5}=\Big)\ 18639+7s$;

Where $s\sqsupset\frac{1}{5}$; And $s\sqsubset-\left(\frac{18639}{7}=\right)2662\frac{5}{7}$.

And therefore, when $z=6$, s has 2663 values.

When $z=7$; Then $5x+7y=(93256-63)=93193$.

Th. $x=\left(\frac{93193-7y}{5}=\right)18638-y+\frac{3-2y}{5}$:

And (proceeding in the ſame manner as when $z=2$) $y=5s-1$;

And $x=\left(\frac{93193-7\times\overline{5s-1}}{5}=\right)18640-7s$:

Where $s\sqsubset\frac{1}{5}$; And $s\sqsupset\left(\frac{18640}{7}=\right)2662\frac{6}{7}$.

When $z=8$; Then $5x+7y=(93256-72=)93184$.

Th. $x=\left(\frac{93184-7y}{5}=\right)18636-y+\frac{4-2y}{5}$;

And (by continuing the proceſs as when $z=3$) $y=2-5s$;

And $x=\frac{93184-7\times\overline{2-5s}}{5}=18634+7s$:

Where $s\sqsupset\frac{2}{5}$; And $s\sqsubset-\left(\frac{18634}{7}=\right)2662$.

When $z=9$; Then $5x+7y=(93256-81=)93175$;

Th. $x=\left(\frac{93175-7y}{5}=\right)18635-y-\frac{2y}{5}$,

Where $y=5s$ (by the ſame ſteps as when $z=4$)

And $x=\left(\frac{93175-7\times5s}{5}=\right)18635-7s$;

Where $s\sqsubset0$; And $s\sqsupset\left(\frac{18635}{7}=\right)2662\frac{1}{7}$.

When $z=10$; Then $5x+7y=(93256-90=)93166$.

Th. $x=\left(\frac{93166-7y}{5}=\right)18633-y+\frac{1-2y}{5}$, Where $y = 5s - 2$ (by arguing as when $z = 5$)

And $x = \left(\frac{93166 - \overline{5s - 2} \times 7}{5} = \right) 18636 - 7s$:

Where $s \sqsubset \frac{2}{5}$; and $s \sqsupset \frac{18636}{7} = 2662\frac{2}{7}$.

From the above operations it is evident, that the ſame equations, expreſſing the value of y, recur every 5th proceſs, 5 being the coefficient of x; and that the correſponding equations, expreſſing the value of x, differ only by (9) the coefficient of z. That is,

					Values.
If $z = 1$;	$y = 1 - 5s$;	$x = 18648 + 7s$;	And $s \sqsupset \frac{1}{5}$.	$s \sqsubset -2664$,	has 2664.
$z = 6$;		$x = 18639 + 7s$;		$s \sqsubset -2662\frac{5}{7}$,	2663.
$z = 11$;		$x = 18630 + 7s$;		$s \sqsubset -2661\frac{3}{7}$,	2662.
$z = 16$;		$x = 18621 + 7s$;		$s \sqsubset -2660\frac{1}{7}$,	2661.
$z = 21$;		$x = 18612 + 7s$;		$s \sqsubset -2658\frac{6}{7}$,	2659.
$z = 26$;		$x = 18603 + 7s$;		$s \sqsubset -2657\frac{4}{7}$,	2658.
$z = 31$;		$x = 18594 + 7s$;		$s \sqsubset -2656\frac{2}{7}$,	2657.
$z = 36$;		$x = 18585 + 7s$;		$s \sqsubset -2655$,	2655.
$z = 41$;		$x = 18576 + 7s$;		$s \sqsubset -2653\frac{5}{7}$,	2654.
$z = 46$;		$x = 18567 + 7s$;		$s \sqsubset -2652\frac{3}{7}$,	2653.
&c.		&c.		&c.	&c.

And the 2072 term of the ſeries will be,

$z = 10356$; $y = 1 - 5s$; $x = 9 + 7s$; $s \sqsubset -1\frac{2}{7}$, has 2 [values.

This whole ſeries, therefore, is compoſed of 7 arithmetical progreſſions, whoſe common difference is 9, and the number of terms $\frac{2072}{7} = 296$ each. Alſo,

					Values.
If $z = 2$;	$y = 5s - 1$;	$x = 18649 - 7s$;	And $s \sqsubset \frac{1}{5}$,	$s \sqsupset 2664\frac{1}{7}$,	s has 2664.
$z = 7$;		$x = 18640 - 7s$;		$s \sqsupset 2662\frac{6}{7}$,	2662.
$z = 12$;		$x = 18631 - 7s$;		$s \sqsupset 2661\frac{4}{7}$,	2661.
$z = 17$;		$x = 18622 - 7s$;		$s \sqsupset 2660\frac{2}{7}$,	2660.
$z = 22$;		$x = 18613 - 7s$;		$s \sqsupset 2659$,	2658.
$z = 27$;		$x = 18604 - 7s$;		$s \sqsupset 2657\frac{5}{7}$,	2657.
$z = 32$;		$x = 18595 - 7s$;		$s \sqsupset 2656\frac{3}{7}$,	2656.
$z = 37$;		$x = 18586 - 7s$;		$s \sqsupset 2655\frac{1}{7}$,	2655.
&c.		&c.		&c.	&c.

And the 2072 term of the ſeries will be,

$z = 10357$; $y = 5s - 1$; $x = 10 - 7s$; $s \sqsupset 1\frac{3}{7}$, s has 1.

And therefore this whole ſeries is compoſed of 7 arithmetical progreſſions, differing from the former, only in ſome of the greateſt terms.

					Values.
If $z = 3$;	$y = 2 - 5s$;	$x = 18643 + 7s$;	$s \sqsupset \frac{2}{5}$.	$s \sqsubset -2663\frac{2}{7}$,	s has 2664.
$z = 8$;		$x = 18634 + 7s$;		$s \sqsubset -2662$,	2662.
$z = 13$;		$x = 18625 + 7s$;		$s \sqsubset -2660\frac{5}{7}$,	2661.
$z = 18$;		$x = 18616 + 7s$;		$s \sqsubset -2659\frac{3}{7}$,	2660.
$z = 23$;		$x = 18607 + 7s$;		$s \sqsubset -2668\frac{1}{7}$,	2659.
$z = 28$;		$x = 11598 + 7s$;		$s \sqsubset -2656\frac{6}{7}$,	2657.
$z = 33$;		$x = 18589 + 7s$;		$s \sqsubset -2655\frac{4}{7}$,	2656.
$z = 38$;		$x = 18580 + 7s$;		$s \sqsubset -2654\frac{2}{7}$,	2655.
&c.		&c.		&c.	&c.

And the 2072 term of this ſeries will be

$z = 10368$; $x = 4 + 7s$; $s \sqsubset -\frac{4}{7}$; has 1.

And, therefore, this whole ſeries is compoſed of ſeven ſuch arithmetical progreſſions, as the former.

					Values.
If $z = 4$;	$y = 5s$;	$x = 18644 - 7s$;	$s \sqsubset 0$;	$s \sqsupset 2663\frac{3}{7}$,	s has 2663.
$z = 9$;		$x = 18635 - 7s$;		$s \sqsupset 2662\frac{1}{7}$,	2662.
$z = 14$;		$x = 18626 - 7s$;		$s \sqsupset 2660\frac{6}{7}$,	2660.
$z = 19$;		$x = 18617 - 7s$;		$s \sqsupset 2659\frac{4}{7}$,	2659.
$z = 24$;		$x = 18608 - 7s$;		$s \sqsupset 2658\frac{2}{7}$,	2658.
$z = 29$;		$x = 18599 - 7s$;		$s \sqsupset 2657$,	2656.
$z = 34$;		$x = 18590 - 7s$;		$s \sqsupset 2655\frac{5}{7}$,	2655.
$z = 39$;		$x = 18581 - 7s$;		$s \sqsupset 2654\frac{3}{7}$,	2654.
&c.		&c.		&c.	&c.

And the 2071 term of this ſeries will be

$z = 10354$; $x = 14 - 7s$; $s \sqsupset 2$, s has 1 value.

But if another term be taken, the value of s will be nothing. For if

$z = 10359$; $x = 5 - 7s$; $s \sqsupset 0\frac{5}{7}$, s has 0.

Therefore the whole ſeries may be conſidered as conſiſting of 7 ſuch arithmetical progreſſions as the former.

Laſtly,

					Values.
If $z = 5$;	$y = 5s - 2$;	$x = 18645 - 7s$;	$s \sqsubset \frac{2}{5}$	$s \sqsupset 2663\frac{4}{7}$,	s has 2663.
$z = 10$;		$x = 18636 - 7s$;		$s \sqsupset 2662\frac{2}{7}$,	2662.
$z = 15$;		$x = 18627 - 7s$;		$s \sqsupset 2661$,	2660.
$z = 20$;		$x = 18618 - 7s$;		$s \sqsupset 2659\frac{5}{7}$,	2659.
$z = 25$;		$x = 18609 - 7s$;		$s \sqsupset 2658\frac{3}{7}$,	2658.
$z = 30$;		$x = 18600 - 7s$,		$s \sqsupset 2657\frac{1}{7}$,	2657.
$z = 35$;		$x = 18591 - 7s$;		$s \sqsupset 2655\frac{6}{7}$,	2655.
$z = 40$;		$x = 18582 - 7s$;		$s \sqsupset 2654\frac{4}{7}$,	2654.
&c.		&c.		&c.	&c.

And the 2071 term of this ſeries will be

$z = 10355$; $x = 15 - 7s$; $s \sqsupset 2\frac{1}{7}$, s has 2 values.

And in the next term s will have no value. And therefore this ſeries conſiſts alſo of ſeven arithmetical progreſſions, ſuch as the former.

Now the ſum of the 7 firſt terms of the progreſſions ariſing when $z=1$, 6, 11, &c. is 18624;
ditto $z=2$, 7, 12, &c. is 18618;
ditto $z=3$, 8, 13, &c. is 18619;
ditto $z=4$, 9, 14, &c. is 18613;
ditto $z=5$, 10, 15, &c. is 18614;
Therefore the ſum of the firſt terms of the 35 progreſſions - - - - - - - } is 93088.

And the whole number of anſwers, which can be given to the queſtion, are comprized in 296 terms of an arithmetical progreſſion, whoſe greateſt term is 93088, and common difference $(35 \times 9 =)$ 315: The ſum of which progreſſion will, upon computation, appear to be 13801148, the ſame as the reſult of the former ſolution.

COROL. I.

As in thoſe equations, which have but two indetermined quantities, it ſeems ſufficient to point out a few anſwers, and to determine their number; ſo in thoſe, where there are three indetermined quantities, the queſtion may be conſidered as ſolved, as ſoon as the arithmetical progreſſions that contain the number of anſwers are aſcertained: Now, although from the different methods, by which the ſolution of thoſe queſtions has been attempted, it appears, that it will be difficult to determine, from the given equation, without ſome kind of proceſs, how many ſuch progreſſions will be neceſſary; yet, from the laſt method, we may ſafely conclude, that their number can never exceed the product of the coefficients of x and y.

COROL. II.

Equations, which contain four indetermined quantities, may be ſolved by the joint application of the methods above delivered: That is to ſay, by ſubſtituting 1, 2, 3, *&c.* for that indetermined quantity which has the greateſt coefficient, and then applying either of the former methods to ſolve the reſulting equations, in which there will be three indetermined quantities, as will be hereaſter exemplified.

Scholium. Having before pointed out the caſes, whoſe ſolutions may be moſt readily come at, in ſuch queſtions which have but two indetermined numbers, they will aſſiſt in finding thoſe which are eaſieſt, when there are three numbers not determined.

Mr. *De Moivre*, in queſtion 6, has aſſumed the coefficient of z to be the ſum of the coefficients of x and y, and therein has given the eaſieſt caſe poſſible, and the following queſtions are introduced to ſhew the effects of aſſuming for the coefficient of z a multiple of the coefficient of x, or of y.

QUESTION IX.

How many different anſwers in affirmative integers, will the equation $3x + 5y + 9z = 1849$, admit of?

SOLUTION.

Here $x = \left(\frac{1849 - 5y - 9z}{3} = \right) 616 - y - 3z + \frac{1-2y}{3}$.

Let $r = \frac{1-2y}{3}$; Then $3r = 1-2y$;

Th. $y = \left(\frac{1-3r}{2} = \right) -r + \frac{1-r}{2}$.

Let $s = \frac{1-r}{2}$; Then $2s = 1-r$;

Th. $r = 1-2s$;

$y = \left(\frac{1-3 \times \overline{1-2s}}{2} = \right) 3s-1$;

And $x = \left(\frac{1849 - 5 \times \overline{3s-1} - 9z}{3} = \right) 618 - 5s - 3z$:

Where $s \sqsubset \frac{1}{3}$, And $s \sqsupset \left(\frac{618-3}{5} = \right) 103$.

Now in this caſe the values of z are all limited by the column which contains the value of x.

Values.

If $s = 1$; $x = 613 - 3z$; $y = 3 - 1 = 2$; z has 204

$s = 2$; $x = 608 - 3z$; $y = 6 - 1 = 5$; z has 202

$s = 3$; $x = 603 - 3z$; $y = 9 - 1 = 8$; z has 200

$s = 4$; $x = 598 - 3z$; $y = 12 - 1 = 11$; z has 199

&c. *&c.* *&c.* *&c.*

Therefore the values of z make three arithmetical progreſſions, 204, 199, 194, *&c.* 202, 197, 192, *&c.* and 200, 195, 190, *&c.* And becauſe $s \sqsupset 103$, each progreſſion will conſiſt of $\left(\frac{102}{3} = \right) 34$ terms.

QUESTION X.

How many different anſwers, in affirmative integers, will the equation $3x + 5y + 20z = 1849$, admit of?

SOLUTION.

Here $x = \left(\frac{1849-5y-20z}{3} =\right) 616 - y - 6z + \frac{1-2y-2z}{3}$;

Let $r = \frac{1-2y-2z}{3}$; Then $3r = 1 - 2y - 2z$;

Th. $y = \left(\frac{1-2z-3r}{2} =\right) -z - r + \frac{1-r}{2}$:

Let $s = \frac{1-r}{2}$; Then $2s = 1 - r$;

Th. $r = 1 - 2s$;

$y = \left(\frac{1-2z-3\times\overline{1-2s}}{2} =\right) 3s - z - 1$;

And $x = \frac{1849 - 15s + 5z + 5 - 20z}{3}$;

Or $x = \left(\frac{1854-15s-15z}{3} =\right) 618 - 5s - 5z$.

Where (becauſe the coefficients of s and z are equal)
Put $p = s + z$, or $s = p - z$:
Then $x = 618 - 5p$;
And $y = (3\times\overline{p-z} - z - 1 =)\ 3p - 4z - 1$:
Where $p \sqsubset \left(\frac{618}{5} =\right) 123\frac{3}{5}$, and $p \sqsupset \left(\frac{5}{3} =\right) 1\frac{2}{3}$,
So that p will have 122 values.

And the values of z are all limitted by the column which contains the values of y.

If $p=2$; $x=618-10=608$; $y=5-4z$; z has 1 value.
$p=3$; $x=618-15=603$; $y=8-4z$; z 1 ditto.
$p=4$; $x=618-20=198$; $y=11-4z$; z 2 values.
$p=5$; $x=618-25=193$; $y=14-4z$; z 3 values.
$p=6$; $x=618-30=188$; $y=17-4z$; z 4 ditto.
$p=7$; $x=618-35=183$; $y=20-4z$; z 4 ditto.
&c. &c. &c. &c.

Here the number of the values of z compoſe 4 arithmetical progreſſions, two of which are equal, *viz.* 1, 4, 7, 10, &c. 1, 4, 7, 10, &c. 2, 5, 8, 11, &c. and 3, 6, 9, 12, &c. And becauſe $\frac{122}{4}=30\frac{2}{4}$, the two firſt will conſiſt of 31 terms; and the two laſt of thirty.

Hence, *Firſt*, In any queſtion of this kind, if either of the leſſer coefficients will meaſure the greater; the number of arithmetical progreſſions, neceſſary to its ſolution, will not exceed the number by which the ſaid leſſer coefficient meaſures the greater.

Secondly, If in any column, A or B, expreſſing the values of x or y, the coefficient of z be greater than the common difference of the progreſſion, whereby the abſolute numbers encreaſe or decreaſe; then two or more of the arithmetical progreſſions, exhibiting the number of the values of z will be equal.

In the following queſtion the coefficients of y and z are ſo aſſumed, that if they be ſeverally divided by the coefficient of x, the ſame remainder may be left.

QUESTION XI.

How many different anſwers, in affirmative integers, will the equation $3x+5y+17z=1849$ admit of?

SOLUTION.

Here $x=\left(\frac{1849-5y-15z}{3}=\right)616-y-5z+\frac{1-2y-2z}{3}$:

Let $r=\frac{1-2y-2z}{3}$; Then $3r=1-2y-2z$;

Th. $y=\left(\frac{1-2z-3r}{2}=\right)-z-r+\frac{1-r}{2}$:

Let $s=\frac{1-r}{2}$; Then $2s=1-r$;

Th. $r=1-2s$;

$y=\left(\frac{1-2z-3\times\overline{1-2s}}{1}=\right)3s-z-1$;

And $x=\frac{1849-15s+5z+5-17z}{3}$

Or $x=\left(\frac{1854-15s-12z}{3}=\right)618-5s-4z$:

Wh. $s \sqsubset \frac{2}{3}$; And $s \sqsupset \left(\frac{614}{5}=\right)122\frac{4}{5}$.

	A		B	
If $s=1$;	$x=613-4z$;	And	$y=2-z$;	z has 1 value.
$s=2$;	$x=608-4z$;		$y=5-z$;	z has 4 values.
$s=3$;	$x=603-4z$;		$y=8-z$;	z has 7 values.
&c.	&c.		&c.	&c.

If $s=122$; $x=8-4z$; And $y=365-z$; z has 1 value.
$s=121$; $x=13-4z$; $y=362-z$; z 3 values.
$s=120$; $x=18-4z$; $y=359-z$; z 4
$s=119$; $x=23-4z$; $y=356-z$; z 5
$s=118$; $x=28-4z$; $y=353-z$; z 6
$s=117$; $x=33-4z$; $y=350-z$; z 8
&c. &c. &c. &c.

Here the column B gives the arithmetical progreſſion 1, 4, 7, 10, &c. only, for the numbers of the values of z; but the column A produces the four progreſſions following, *viz.* 1, 6, 11, 16, &c. 3, 8, 13, 18, &c. 4, 9, 14, 19, &c. And 5, 10, 15, 20, &c.

Hence, when the two greater coefficients, being ſeverally divided by the leſſer, leave the ſame remainder; then one of the columns will exhibit a ſingle progreſſion for the numbers of the values of z, while the ſame are limitted thereby.

QUESTION XII.

It is required to find what number of different anſwers, in affirmative integers, may be given to the following equation, *viz.* $2x+3y+5z+30u=100003$.

SOLUTION.

Here (following the method uſed in the ſecond ſolution of queſtion 8).

Let $u=1$; Then $2x+3y+5z+30=100003$;
Th. $2x=100003-30-3y-5z$,

Or $x = \left(\frac{99973-3y-5z}{2} = \right) 49986 - y - 2z + \frac{1-y-z}{2}$:

Let $r = \frac{1-y-z}{2}$; Then $2r = 1 - y - z$;

Th. $y = 1 - z - 2r$;

And $x = \frac{99973 - 3 \times \overline{1 - z - 2r} - 5z}{2}$,

Or $x = \left(\frac{99973 - 3 + 3z + 6r - 5z}{2} = \right) 49985 + 3r - z$.

In the expreſſion $y = 1 - z - 2r$, it is evident, that if r be nothing or affirmative, y will be negative: Let therefore the ſign of r be changed in the values of both x and y; And then,

$y = 1 + 2r - z$,

And $x = 49985 - 3r - z$;

Th. $r \sqsubset (1 - 1 =) 0$; And $r \sqsupset \left(\frac{49984}{3} = \right) 16661\frac{1}{3}$.

	A	B	
If $r = 1$;	$x = 49982 - z$;	$y = 3 - z$;	z has 2 values.
$r = 2$;	$x = 49979 - z$;	$y = 5 - z$;	z 4
$r = 3$;	$x = 49976 - z$;	$y = 7 - z$;	z 6
&c.	&c.	&c.	&c.
$r = 16661$;	$x = 2 - z$;	$y = 33323 - z$;	has 1 value.
$r = 16660$;	$x = 5 - z$;	$y = 33321 - z$;	4 values.
$r = 16659$;	$x = 8 - z$;	$y = 33319 - z$;	7
&c.	&c.	&c.	&c.

Now, in order to find the number of terms in which the value of z is limitted by the column B,

Let $49982 - 3n = 3 + 2n$,

And $5n = 49979$; Th. $n = 9995\frac{4}{5}$.

Therefore the column B limits the value of z in 9996 terms; and the column A in $(16661 - 9996 =)$ 6665 terms.

Let $u = 2$; Then $2x + 3y + 5z + 60 = 100003$;
Th. $2x = 100003 - 60 - 3y - 5z$,
Or $x = \left(\frac{99943 - 3y - 5z}{2} = \right) 49971 - 2 - y + \frac{1 - y - z}{2}$;

Which remainder being the ſame as before obtained,

Th. $\begin{cases} y = 1 + 2r - z; \text{ as in the former ſubſtitution,} \\ x = 49970 - 3r - z, \text{ differing from the former by } 15. \end{cases}$

$$r \sqsubset 0; \text{ And } r \sqsupset \left(\frac{49969}{3} = \right) 16656\tfrac{1}{3}.$$

Now when $r = 1$, or 2, or 3, &c. z has 2, or 4, or 6 values, as before, and if

$r = 16656$; $x = 2 - z$; $y = 33313 - z$; has 1 value.
$r = 16655$; $x = 5 - z$; $y = 33311 - z$; has 4 values.
$r = 16654$; $x = 8 - z$; $y = 33309 - x$; has 7 ditto.
&c. &c. &c. &c.

Alſo to find when the column B ceaſes to limit the value of z,

$$49967 - 3n = 3 + 2n;$$

Th. $5n = 49964$, And $n = \frac{49964}{5} = 9992\tfrac{4}{5}$.

Therefore the column B limits the values of z in 9993 terms; and the column A in $(16656 - 9993 =)$ 6663 terms, If the above kinds of proceſs be continued, by ſubſtituting 3, 4, 5, *&c.* for *u*, it will appear that when

$u=1$;	The number of the values of *z* will be two arithmetical progreſſions, the firſt of which will have 2 for its leaſt term and common difference, and will conſiſt of	9996	Terms; and the other will have one for its leaſt term, and 3 for its common difference; and will conſiſt of	6665	Terms.
$u=2$;		9993		6663	
$u=3$;		9990		6661	
$u=4$;		9987		6659	
$u=5$;		9984		6657	
$u=6$;		9981		6655	
&c.		*&c*		*&c.*	

Or aſſuming *u*, according to its other limit,

$$u \sqsubset \left(\frac{100003-2-3-5}{30}=\frac{99993}{30}=\right)3333\tfrac{3}{30};$$

Then the two arithmetical progreſſions which contain the number of the values of *z* will, when

$u=3333$	be	0	Terms of the progreſſion 2, 4, 6, 8, *&c.* And	1	Terms of the progreſſion 1, 4, 7, 10, *&c.*
$u=3332$		3		3	
$u=3331$		6		5	
$u=3330$		9		7	
$u=3329$		12		9	
$u=3328$		15		11	
$u=3327$		18		13	
&c.		*&c.*		*&c.*	

Which may be concluded, without making the ſubſtitutions, by finding the 3333*d*, 3332*d*, &c. terms of the above given progreſſions.

Now the ſum of	Terms of the arithmetical progreſſion.	
0	$2+4+6;$	$=0;$
3	$2+4+6;$	$=12;$
6	$2+4+6+8+10+12;$	$=12+30;$
9	$2+4+6+8+10$, *&c.*	$=12+30+48;$
12	$2+4+6+8+10$, *&c.*	$=12+30+48+66;$
15	$2+4+6+8+10$, *&c.*	$=12+30+48+56+84$
&c.	*&c.*	*&c.*

And the ſums are a rank of numbers, whoſe ſecond differences are equal, as appears below.

Sums.	1ſt Differences.	2d Differences.
0.		
12.	12.	
12+30.	30.	18.
12+30+48.	48.	18.
12+30+48+66.	66.	18.
12+30+48+66+84.	84.	18.
&c.	&c.	&c.

And the ſum of 3333 terms of a ſeries, whoſe firſt term is 0, the firſt of its firſt differences 12, and the firſt of its ſecond differences 18, will (by Queſt. 52. Part II. Vol. I.) be

$$3333\times0+\frac{3333\times3332\times12}{1\times2}+\frac{3333\times3332\times3331\times18}{1\times2\times3}.$$

Again the ſum of	Terms of the arithm. progr.	
1	1;	=1;
3	1+3+5;	=1+8;
5	1+3+5+7+9;	=1+8+16;
7	1+3+5+7+9,&c.	=1+8+16+24;
9	1+3+5+7+9,&c.	=1+8+16+24+32;
&c.	&c.	&c.

In which the ſums are	1ſt Differences.	2d Differences.
1.		
1+8.	8.	
1+8+16.	16.	8.
1+8+16+24.	24.	8.
1+8+16+24+32.	32.	8.

And the ſum of 3333 terms, thereof, will be

$$3333\times1+\frac{3333\times3332\times8}{1\times2}+\frac{3333\times3332\times3331\times8}{1\times2\times3}.$$

And, becauſe the number of the terms of both ſeries is the ſame, the ſum of them both will be, $3333\times\overline{0+}$

$$+\frac{3333\times3332}{1\times2}\times\overline{12+8}+\frac{3333\times3332\times3331}{1\times2\times3}\times\overline{18+8};$$

That is 160190378249.

From the above proceſs, it appears that it would have been more convenient to have began the ſubſtitution with the greateſt value of n; becauſe then, the leaſt terms of the arithmetic ſeries, whoſe ſums are the anſwer, would have been immediately produced.

QUESTION XIII.

Let there be a ſeries formed in the following manner, *viz.* Let the firſt term of it be the ſum of n numbers in arithmetical progreſſion; the ſecond term, the ſum of $n+m$ numbers; the third term, the ſum of $n+2m$ numbers; the fourth term, the ſum of $n+3m$ numbers of the ſame progreſſion, *&c.* then the ſum of p terms of this ſeries is required?

SOLUTION.

Let the given arithmetical progreſſion be repreſented by a, $a+d$, $a+2d$, $a+3d$, *&c.* Then will

n terms of a, $a+d$, *&c.* be $na+\frac{nn-n}{2}d$;

$n+m$ terms thereof $na+ma+\frac{nn+2nm+mm-n-m}{2}d$;

$n+2m$ terms $na+2ma+\frac{nn+4nm+4mm-n-2m}{2}d$;

$n+3m$ terms $na+3ma+\frac{nn+6nm+9mm-n-3m}{2}d$,

&c. &c. &c.

The firſt differences of which are	And their ſecond differences.
$ma+\frac{2nm+mm-m}{2}d.$	
$ma+\frac{2nm+3mm-m}{2}d.$	$mmd.$
$ma+\frac{2nm+5mm-m}{2}d.$	$mmd.$
&c. &c.	&c.

And therefore (by Queſt. 52. Part II. Vol. I.) the ſum of p terms of the ſeries will be,

$$p\times\overline{na+\frac{nn-n}{2}d}+\frac{p\cdot\overline{p-1}}{2}\times\overline{ma+\frac{2nm+mm-m}{2}d}.$$

$$\left(+\frac{p\cdot\overline{p-1}\cdot\overline{p-2}}{1\cdot2\cdot3\cdot}mmd,\right.$$

Or $\left(\text{if } P=\frac{\overline{p-1}\times m}{2}\right)$ it will become

$$\overline{\overline{n+P}\times a+\overline{\frac{2n+m-1}{2}+\frac{\overline{p-2}\times m}{3}}\times P+\frac{n\cdot\overline{n-1}}{1\cdot2}\times d}\times p.$$

EXAMPLE. If the arithmetical progreſſion be 3, 5, 7, 9, 11, 13, &c. and therefrom be formed a ſeries, 24, 48, 80, 120, 168, 224, 288, &c. by taking the ſum of the four firſt numbers for the firſt term, and for the remaining terms the ſum of 6, 8, 10, 12, *&c.* numbers of the ſame progreſſion. Then let the ſum of 7 terms of this ſeries be required.

Here $a=3$; $d=2$; $n=4$; $m=2$ and $p=7$;

Then $\frac{6\times 2}{2}=6=P$; $\overline{4+6}\times 3=30$; $\frac{2\times 4+2-1}{2}=$

$\frac{\overline{7-2}\times 2}{3}=\frac{10}{3}$; $\frac{4\times 3}{2}=6$;

$\overline{\frac{9}{2}+\frac{10}{3}}\times 6=(27+20=)\ 47$; And $47+6=53$;

$53\times 2=106$; And $106+30=136$; Laſtly $136\times 7=952$ The ſum required.

QUESTION XIV.

How many different values can x, y, z and u have, in affirmative integers, in the equation $3x+5y+19z+143u=91306$?

Method of SOLUTION.

Here $u \sqsubset 638\frac{45}{143}$.

If $u=638$; Then $3x+5y+19z+91324=91306$

Or $3x+5y+19z=72$;

Whence $x=\left(\frac{72-5y-19z}{3}=\right)24-y-6z-\frac{2y+z}{3}$.

Let $r=\frac{2y+z}{2}$; Then $2y=3r-z$;

Th. $y = \left(\frac{3r-z}{2} =\right) r + \frac{r-z}{2}.$

Let $s = \frac{r-z}{2}$; And $2s = r - z$;

Th. $r = 2s + z$,

$$y = \left(\frac{3 \times \overline{2s+z} - z}{2} = \frac{6s + 3z - z}{2} =\right) 3s + z;$$

And $x = 24 - 5s - 8z$;

And putting $s = p - z$; Then $y = 3p - 2z$; Where $p \sqsubset \frac{2}{3}$,

And $x = 24 - 5p - 3z$; Where $p \sqsupset \left(\frac{24-3}{5} = \frac{21}{5} =\right) 4\frac{1}{5}$.

$$\left.\begin{array}{llll} \text{If } p=1; & \text{Then } x=19-3z; & \text{And } y=3-2z; & z \text{ has } 1 \\ p=2; & x=14-3z; & y=6-2z; & z \quad 2 \\ p=3; & x=9-3z; & y=9-2z; & z \quad 2 \\ p=4; & x=4-3z; & y=12-2z; & z \quad 1 \end{array}\right\} \text{Values.}$$

Where the values of z are limitted by each of the columns A and B for two terms.

If $u = 637$; Then $3x + 5y + 19z + 91091 = 91306$,
Or $3x + 5y + 19z =; 215$;

Whence $x = \left(\frac{215 - 5y - 19z}{3} =\right) 71 - y - 6z + \frac{2 - 2y - z}{3}.$

Let $r = \frac{2 - 2y - z}{3}$; Then $2y = 2 - z - 3r$,

Th. $y = \left(\frac{2 - z - 3r}{2} =\right) 1 - r - \frac{z+r}{2}$:

Let $s = \frac{z+r}{2}$; Then $2s = z + r$; And $2s - z = r$;

Then $y = \left(\frac{2 - z - 3 \times \overline{2s - z}}{2} =\right) 1 + z - 3s$, And $x = \left(\dfrac{215 - 5 \times \overline{1+z-3s} - 19z}{3}\right) 70 + 5s - 8z.$

Wherein (putting $s = \overline{z-p}$) there ariſes

$$y = (1 + z - 3 \times \overline{z-p} =)\ 3p + 1 - 2z,$$

And $x = (70 + 5 \times \overline{z-p} - 8z =)\ 70 - 5p - 3z;$

Where $p \sqsubset \left(\dfrac{2 \times 1 - 1}{3} =\right) \frac{1}{3}$; And $p \sqsupset 13\frac{2}{5}$.

$$\left.\begin{array}{llll}
\text{If } p = 1; & x = 65 - 3z; & \text{And } y = 4 - 2z; & z \text{ has } 1 \\
p = 2; & x = 60 - 3z; & y = 7 - 2z; & z \text{ has } 3 \\
p = 3; & x = 55 - 3z; & y = 10 - 2z; & z \text{ has } 4 \\
p = 4; & x = 50 - 3z; & y = 13 - 2z; & z \text{ has } 6 \\
p = 5; & x = 45 - 3z; & y = 16 - 2z; & z \text{ has } 7 \\
p = 6; & x = 40 - 3z; & y = 19 - 2z; & z \text{ has } 9 \\
p = 7; & x = 35 - 3z; & y = 22 - 2z; & z \text{ has } 10 \\
p = 8; & x = 30 - 3z; & y = 25 - 2z; & z \text{ has } 9 \\
p = 9; & x = 25 - 3z; & y = 28 - 2z; & z \text{ has } 8 \\
p = 10; & x = 20 - 3z; & y = 31 - 2z; & z \text{ has } 6 \\
p = 11; & x = 15 - 3z; & y = 34 - 2z; & z \text{ has } 4 \\
p = 12, & x = 10 - 3z; & y = 37 - 2z; & z \text{ has } 3 \\
p = 13; & x = 5 - 3z; & y = 40 - 2z; & z \text{ has } 1
\end{array}\right\} \text{Values.}$$

Where the values of z are limitted by the column B for 7 terms; and by the column A for the remaining 6 terms.

If u 636; Then $3x + 5y + 19z + 90948 = 91306$,

Or $3x + 5y + 19z = 358$;

Whence $x = \left(\dfrac{358 - 5y - 19z}{3} =\right) 119 - y - 6z + \dfrac{1 - 2y - z}{3}$

Let $r = \frac{1-2y-z}{3}$; Then $2y = 1 - z - 3r$;

Th. $y = \left(\frac{1-z-3r}{2} =\right) -r + \frac{1-z-r}{2}$:

Let $s = \frac{1-z-r}{2}$; Then $2s = 1 - z - r$,

And $r = 1 - z - 2s$:

Now $y = \left(\frac{1-z-3\times\overline{1-z-2s}}{2} =\right) \frac{1-z-3+3z+6s}{2}$,

Or $y = \left(\frac{6s+2z-2}{2} =\right) 3s + z - 1$;

Laſtly $x = \frac{358-15s-5z+5-19z}{3}$,

Or $x = \left(\frac{363-15s-24z}{3} =\right) 121 - 5s - 8z$.

Where, becauſe z has different ſigns in the values of x and y, let us ſubſtitute $s = p - z$;

Then $x = (121 - 5\times\overline{p-z} - 8z =)\ 121 - 5p - 3z$;
And $y = (3\times\overline{p-z} + z - 1 =)\ 3p - 2z - 1$:
Where $p \sqsubset \left(\frac{2\times1+1}{3} = \frac{3}{3} =\right) 1$; And $p \sqsupset 23\frac{3}{5}$;
And there are 22 values of p in the whole.

$$\left.\begin{array}{llll} \text{If } p=2; & \text{Then } x=111-3z; & \text{And } y=5-2z; & z \text{ has } 2 \\ p=3; & x=106-3z; & y=8-2z; & z \text{ has } 3 \\ p=4; & x=101-3z; & y=11-2z; & z \text{ has } 5 \end{array}\right\} \text{Values.}$$

&c. &c. &c. &c.

$$\left.\begin{array}{llll}\text{If } p=23;\ \text{Then } x=\ 6-3z; & \text{And } y=68-2z; & z \text{ has } 1 \\ \quad p=22; \qquad\quad x=11-3z; & \quad\ y=65-2z; & z \text{ has } 3 \\ \quad p=21; \qquad\quad x=16-3z; & \quad\ y=62-2z; & z \text{ has } 5 \\ \quad p=20; \qquad\quad x=21-3z; & \quad\ y=59-2z; & z \text{ has } 6 \\ \quad \&c. \qquad\qquad \&c. & \quad\ \&c. & \&c. \end{array}\right\}\text{Values.}$$

Alſo $\frac{111-5n}{3}=\frac{5+3n}{2}$; Or $222-10n=15+9n$,

Or $222-15=10n+9n$; Th. $n=\left(\frac{207}{19}=\right)10\frac{17}{19}$;

Therefore the values of z are limited by the column B for the firſt 11 terms, and by the column A in the laſt.

If $\quad u=635$, Then $3x+5y+19z+90805=91306$,
Or $\quad 3x+5y+19z=501$;

Whence $x=\left(\frac{501-5y-19z}{3}\right)=167-y-6z-\frac{2y+z}{3}$.

And, by proceeding in the ſame manner as when u was aſſumed $=638$, it will appear

That $\quad y=3s+z$;

And $\quad z=\left(\frac{501-5\times\overline{3s+z}-19z}{3}=\right)167-5s-8z$:

Or, putting $s=p-z$,
$y=3p-2z$; Where $p\sqsubset\frac{z}{3}$;

And $\quad x=167-5p-3z$; And $p\sqsupset\left(\frac{167-3}{5}=\right)32\frac{4}{5}$.

Here it is evident, that the value of y (and the leſſer limit of p which depends thereon) are the ſame with thoſe found when $u=638$; Alſo that the value of x, here found, is greater than the value of x, when $u=638$, by 143, the coefficient of u; And conſequently the greater limit of p exceeds the former, by $\left(\frac{143}{5}=\right)$ $28\frac{3}{5}$.

$$\left.\begin{array}{llll} \text{If } p=1; & \text{Then } x=162-3z; & \text{And } y=3-2z; & z \text{ has } 1 \\ p=2; & x=157-3z; & y=6-2z; & z \text{ has } 2 \\ p=3; & x=152-3z; & y=9-2z; & z \text{ has } 4 \\ \&c. & \&c. & \&c. & \&c. \end{array}\right\} \text{Values.}$$

$$\left.\begin{array}{llll} \text{If } p=32; & \text{Then } x=7-3z; & \text{And } y=96-2z; & z \text{ has } 2 \\ p=31; & x=12-3z; & y=93-2z; & z \text{ has } 3 \\ p=30; & x=17-3z; & y=90-2z; & z \text{ has } 5 \\ p=29; & x=22-3z; & y=87-2z; & z \text{ has } 7 \\ \&c. & \&c. & \&c. & \&c. \end{array}\right\} \text{Values.}$$

Hence, ſo long as the values of z are limited by the column B, the arithmetical progreſſions, expreſſing thoſe values, are the ſame with thoſe which aroſe when $u=638$, the number of terms excepted.

Now $\frac{162-5n}{3}=\frac{3+3n}{2}$; Or $324-10n=9+9n$;

Or $324-9=10n+9n$; Th· $n=\left(\frac{315}{19}=\right)16\frac{11}{19}$.

In this caſe, therefore, the values of z are limited by the column B for 17 terms; and for the remaining $(32-17=)$ 15 terms, by the column A.

From this laſt proceſs, (wherein 635 the value of u differs from 638, the firſt aſſumed value thereof, by 3 the coefficient of x) it may be ſafely concluded, that when u is 632, 629, 626, &c. the expreſſion of the value of y will always be $3p-2z$, as in the two aſſumptions above quoted; And the expreſſions of the value of x will conſtantly differ by 143, the coefficient of u.

Hence,

Hence, when $u = 632$; Then

$$\left.\begin{array}{llll} \text{If } p=1\,; & x=305-3x\,; & \text{And } y=3-2z\,; & z \text{ has } 1 \\ p=2\,; & x=300-3z\,; & y=6-2z\,; & z \text{ has } 2 \\ p=3\,; & x=295-3z\,; & y=9-2z\,; & z \text{ has } 4 \end{array}\right\}\text{Values.}$$

&c. &c. &c. &c.

When $u = 629$,

$$\left.\begin{array}{llll} \text{If } p=1\,; & x=448-3z\,; & \text{And } y=3-2z\,; & z \text{ has } 1 \\ p=2\,; & x=443-3z\,; & y=6-2z\,; & z \text{ has } 2 \\ p=3\,; & x=438-3z\,; & y=9-2z\,; & z \text{ has } 4 \end{array}\right\}\text{Values.}$$

&c. &c. &c. &c.

And, therefore, in all the aſſumptions of *u*, that ar contained in the ſeries 638, 635, 632, 629, &c. the ſame arithmetical progreſſions ſhew how many values of *z* are poſſible, when thoſe values are limited by the column B; *viz.* The progreſſions 1, 4, 7, 10, &c, and 2, 5, 8, 11, &c. But the number of the terms in each progreſſion are different in every particular ſubſtitution; Thus, when $u = 638$, there was but one term of each progreſſion uſeful; when $u = 638$, then, becauſe the column B limits the values of *z* for 17 terms; therefor 9 terms of the firſt progreſſion, and 8 terms of the ſecon are uſeful.

It remains to find the number of terms that will b uſeful, in the ſucceeding ſubſtitutions:

When $u=632$, Then $\frac{305-5n}{3}=\frac{3+3n}{2}$;

Th. $610-10n=9+9n$.

Or $610-9=10n+9n$;

Th. $\frac{601}{19}=n$;

That is $31\frac{12}{19}=n$.

Therefore the column B will limit the values of z for 31 terms, after the firſt, that is for 32 terms; and, conſequently, there will be 16 terms of each progreſſion uſeful.

By comparing the laſt proceſs, with that uſed when 635 was ſubſtituted for u; it will appear, that the values of n differ by $\left(\frac{143\times2}{19}=\frac{286}{19}=\right)15\frac{1}{19}$; which will alſo be the caſe in all ſucceeding ſubſtitutions for u, that differ by 3.

Therefore when

		And the number of the terms, for which the column B limits the values of z, will be	Therefore	Terms of the progression, 1, 4, 7, 10, &c. And	Terms of the progression, 2, 5, 8, 11, &c, will be useful.
$u=638$;	$n=1\frac{10}{19}$;	2;		1	1
$u=635$;	$n=16\frac{11}{19}$;	17;		9	8
$u=632$;	$n=31\frac{12}{19}$;	32;		16	16
$u=629$;	$n=46\frac{13}{19}$;	47;		24	23
$u=626$;	$n=61\frac{14}{19}$;	62;		31	31
$u=623$;	$n=76\frac{15}{19}$;	77;		39	38
$u=620$;	$n=91\frac{16}{19}$;	92;		46	46
$u=617$;	$n=106\frac{17}{19}$;	107;		54	53
$u=614$;	$n=121\frac{18}{19}$;	122;		61	61
$u=611$;	$n=137$;	138;		69	69
$u=608$;	$n=152\frac{1}{19}$;	153;		77	76
$u=605$;	$n=167\frac{2}{19}$;	168;		84	84
$u=602$;	$n=182\frac{3}{19}$;	183;		92	91
$u=599$;	$n=197\frac{4}{19}$;	198;		99	99
$u=596$;	$n=212\frac{5}{19}$;	213;		107	106
$u=593$;	$n=227\frac{6}{19}$;	228;		114	114
$u=590$;	$n=242\frac{7}{19}$;	243;		122	121
$u=587$;	$n=257\frac{8}{19}$;	258;		129	129
$u=584$;	$n=272\frac{9}{19}$;	273;		137	136
$u=581$;	$n=287\frac{10}{19}$;	288;		144	144
&c.	&c.	&c.		&c.	&c.

Hence, when the numbers, ſubſtituted for u, differ by $(3\times19=)$ 57, then the number of uſeful terms, in each of the arithmetical progreſſions, which expreſs the values of z, will differ by 143, the coefficient of u.

That is to ſay, when the terms of the ſeries, 638, 581, 524, 467, &c. are ſeverally ſubſtituted for u, Then both the arithmetical progreſſions, expreſſing the values of z, will contain 1, 144, 287, 430, &c. terms; when the terms of the ſeries 635, 578, 521, 464, &c. are ſeverally ſubſtituted for u, then the arithmetical progreſſion, 1, 4, 7, 10, &c. muſt be continued to 9, 152, 295, 438, &c. terms; and the other progreſſion 2, 5, 8, 11, &c. to 8, 151, 294, 437, &c. terms; and ſo on for the reſt.

Now, becauſe $\left(\frac{629}{57}=\right)11\frac{2}{57}$, the four firſt of thoſe pairs of ſeries conſiſt of 11 terms, and the other fifteen of 10 terms each, all which may be ſeparately ſummed by Queſt. 13.

Or let the reſpective terms of the two arithmetical progreſſions be added together, and they will compoſe the progreſſion 3, 9, 15, 21, *&c.* Of which the number of uſeful terms, in each of the ſubſtitutions for *u*, viz. 638, 635, 632, 629, *&c.* will be half the number of terms, for which the column B limits the values of *z*, viz. 1, $\frac{17}{2}$, 16, $\frac{47}{2}$, *&c.* Or, becauſe theſe numbers depend upon the values of *n*, which have been proved to differ by $\left(15\frac{1}{19}\text{ or}\right)\frac{286}{19}$, we may take the half of that, *viz.* $\frac{143}{19}$, for the difference of the number of uſeful terms in each ſubſtitution.

Whence, all the anſwers to the queſtion, which can happen when the terms of the ſeries 638, 635, 632, *&c.* are ſeverally ſubſtituted for *u*, and the number of the values of *z* are limited by the column B, will nearly conſiſt of a ſeries formed from the arithmetical progreſſion 3, 9, 15, 21, *&c.* by taking the firſt term of that progreſſion, for the firſt term of the ſeries; the ſum of $1+\frac{143}{19}$ terms of that progreſſion, for the ſecond term of the ſeries; the ſum of $1+2\times\frac{143}{19}$ terms of that progreſſion, for the third term of the ſeries, *&c.* which ſeries is ſummable by Queſt. 13.

If $u=634$; Then $3x+5y+19z+90662=91306$,

Or $3x+5y+19z=644$;

Whence $x=\left(\frac{644-5y-19z}{3}=\right)214-y-6z+\frac{2-2y-z}{3}$;

Therefore, (by proceeding as when 637 was ſubſtituted for u) $y=1+z-3s$; And $x=213+5s-8z$:

Or (putting $s=z-p$)

$y=3p+1-2z$; And $x=213-5p-3z$:

Where $p \sqsubset \frac{1}{3}$ And $p \sqsupset \left(\frac{210}{5}=\right)42$.

And here the value of y is the ſame, with that found when 637 was ſubſtituted for u, and the value of x exceeds its value, there found, by 143, in the ſame manner, as was obſerved before, concerning the reſults, when the numbers 638 and 635 were, ſeverally, ſubſtituted for u.

$$\left.\begin{array}{llll} \text{If } p=1; & x=208-3z; & y=4-2z; & z \text{ has } 1 \\ p=2; & x=203-3z; & y=7-2z; & z \text{ has } 3 \\ p=3; & x=198-3z; & y=10-2z; & z \text{ has } 4 \end{array}\right\}\text{Values.}$$

&c. &c. &c. &c.

$$\left.\begin{array}{llll} p=41; & x=8-3z; & y=124-2z; & z \text{ has } 2 \\ p=40; & x=13-3z; & y=121-2z; & z \text{ has } 4 \\ p=39; & x=18-3z; & y=118-2z; & z \text{ has } 5 \\ p=38; & x=23-3z; & y=115-2z; & z \text{ has } 7 \end{array}\right\}\text{Values.}$$

&c. &c. &c. &c.

And therefore, ſo long as the values of z are limited by the column B, the arithmetical progreſſions, expreſſing thoſe values, are the ſame with thoſe which aroſe when $u=637$.

Now $\frac{208-5n}{3}=\frac{4+3n}{2}$; Or $416-10n=12+9n$;

Or $416-12=10n+9n$; Th. $n=\frac{404}{19}=21\frac{5}{19}$.

In this caſe, therefore, the values of z are limited by the column B for 22 terms, and by the column A for the remaining ($41-22=$) 19 terms.

Hence, therefore, we may conclude, as before, that in all the ſubſtitutions, for u, that are contained in the ſeries 637, 634, 631, 628, &c. the ſame arithmetical progreſſion will ſhew how many values of z are poſſible, while the number of thoſe values is limited by the column B; and that the number of terms uſeful in each progreſſion will differ by $\frac{286}{19}$. Therefore, when

		And the number of the terms for which the column R limits the values of z, will be		Therefore		Terms of the progreſſion, 1, 4, 7, 10, &c. And		Terms of the progreſſion, 3, 6, 9, 12, &c. will be uſeful.
$u=637$;	$n=6\frac{4}{19}$;		7;		4		3	
$u=634$;	$n=21\frac{5}{19}$;		22:		11		11	
$u=631$;	$n=36\frac{6}{19}$;		37;		19		18	
$u=628$;	$n=51\frac{7}{19}$;		52;		26		26	
$u=625$;	$n=66\frac{8}{19}$;		67;		34		33	
$u=622$;	$n=81\frac{9}{19}$;		82;		41		41	
$u=619$;	$n=96\frac{10}{19}$;		97;		49		48	
$u=616$;	$n=111\frac{11}{19}$;		112;		56		56	
$u=613$;	$n=126\frac{12}{19}$;		127;		64		63	
$u=610$;	$n=141\frac{13}{19}$;		142;		71		71	
$u=607$;	$n=156\frac{14}{19}$;		157;		79		78	
$u=604$;	$n=171\frac{15}{19}$;		172;		86		86	
$u=601$;	$n=186\frac{16}{19}$;		187;		94		93	
$u=598$;	$n=201\frac{17}{19}$;		202;		101		101	
$u=595$;	$n=216\frac{18}{19}$;		217;		109		108	
$u=592$;	$n=232$;		233;		117		116	
$u=589$;	$n=247\frac{1}{19}$;		248;		124		124	
$u=586$;	$n=262\frac{2}{19}$;		263;		132		131	
$u=583$;	$n=277\frac{3}{19}$;		278;		139		139	
$u=580$;	$n=292\frac{4}{19}$;		293;		147		146	
&c.	&c.		&c.		&c.		&c.	

Whence the number of anſwers, of which the queſtion is capable, when the terms of the ſeries 637, 634, &c. are ſeverally ſubſtituted for u, and the number of the values of z are limited by the column B, may be obtained, either exactly, or by approximation, in the manner above ſhewn.

$$\text{If } u = 633; \text{ Then } 3x + 5y + 19z + 90519 = 91306,$$
$$\text{Or } 3x + 5y + 19z = 787.$$

$$\text{Where } x = \left(\frac{787 - 5y - 19z}{3} = \right) 262 - y - 6z + \frac{1 - 2y - z}{3};$$

Whence (proceeding as when 636 was ſubſtituted for u)

$$x = 264 - 5s - 8z; \text{ And } y = 3s + z - 1 : \text{Or (if } s = p - z)$$
$$x = 264 - 5p - 3z; \text{ And } y = 3p - 2z - 1;$$

$$\text{Where } p \sqsubset 1; \qquad \text{And } p \sqsupset \left(\frac{261}{5} = \right) 52\tfrac{1}{5}.$$

And p will have 51 values; Therefore,

$$\left.\begin{array}{llll} \text{If } p = 2; & x = 254 - 3z; & y = 5 - 2z; & z \text{ has } 2 \\ p = 3; & x = 249 - 3z; & y = 8 - 2z; & z \text{ has } 3 \\ p = 4; & x = 244 - 3z; & y = 11 - 2z; & z \text{ has } 5 \\ \&c. & \&c. & \&c. & \&c. \end{array}\right\} \text{Values.}$$

$$\left.\begin{array}{llll} p = 52; & x = 4 - 3z; & y = 155 - 2z; & z \text{ has } 1 \\ p = 51; & x = 9 - 3z, & y = 152 - 2z; & z \text{ has } 2 \\ p = 50; & x = 14 - 3z; & y = 149 - 2z; & z \text{ has } 4 \\ p = 49; & x = 19 - 3z; & y = 146 - 2z; & z \text{ has } 6 \\ \&c. & \&c. & \&c. & \&c. \end{array}\right\} \text{Values.}$$

And here, it does not only happen, as before, that, ſo long as the values of z are limited by the column B, the arithmetical progreſſions, expreſſing thoſe values, are the ſame with thoſe found when $u = 636$; but alſo, that when they are limited by the column A, then the progreſſions, expreſſing thoſe values, will be the ſame with thoſe found when $u=638$, which number (638) exceeds the preſent ſubſtitution (633) by 5, which is the coefficient of y, and the common difference of the numbers in the column A. Hence,

Firſt, we may determine the remainder of the values of z, when limited by the column B; in the ſame manner as before, *viz.*

If $u=636$;	$n=10\frac{17}{19}$;	And the number of the terms for which the column B limits the values of z, will be	11;	Therefore	6	Terms of the progreſſion, 2, 5, 8, 11, &c. And	5	Terms of the progreſſion, 3, 6, 9, 12, &c. will be uſeful.
$u=633$;	$n=25\frac{18}{19}$;		26;		13		13	
$u=630$;	$n=41$;		42;		21		21	
$u=627$;	$n=56\frac{1}{19}$;		57;		29		28	
$u=624$;	$n=71\frac{2}{19}$;		72;		36		36	
$u=621$;	$n=86\frac{3}{19}$;		87;		44		43	
$u=618$;	$n=101\frac{4}{19}$;		102;		51		51	
$u=615$;	$n=116\frac{5}{19}$;		117;		59		58	
$u=612$;	$n=131\frac{6}{19}$;		132;		66		66	
$u=609$;	$n=146\frac{7}{19}$;		147;		74		73	
$u=606$;	$n=161\frac{8}{19}$;		162;		81		81	
$u=603$;	$n=176\frac{9}{19}$;		177;		89		88	
$u=600$;	$n=191\frac{10}{19}$;		192;		96		96	
$u=597$;	$n=206\frac{11}{19}$;		207;		104		103	
$u=594$;	$n=221\frac{12}{19}$;		222;		111		111	
$u=591$;	$n=236\frac{13}{19}$;		237;		119		118	
$u=588$;	$n=251\frac{14}{19}$;		252;		126		126	
$u=585$;	$n=266\frac{15}{19}$;		267;		134		133	
$u=582$;	$n=281\frac{16}{19}$;		282;		141		141	
$u=579$;	$n=296\frac{17}{19}$;		297;		149		148	
&c.	&c.		&c		&c		&c.	

And hence, the number of anſwers of which the queſtion is capable, when the terms of the ſeries, 636, 633, &c. are ſeverally ſubſtituted for u, and the number of the values of z are limited by the column B, may be determined as before.

In order to determine the number of anſwers, when the values of z are limited by the column A, we muſt firſt find how many values p is capable of. When

$u=638$;	$p \sqsubset \frac{2}{3}$;	$p \sqsupset 4\frac{1}{5}$;	And p has	4
$u=637$;	$p \sqsubset \frac{1}{3}$;	$p \sqsupset 13\frac{2}{5}$;	And p has	13
$u=636$;	$p \sqsubset 1$;	$p \sqsupset 22\frac{3}{5}$;	And p has	22
$u=635$;	$p \sqsubset \frac{2}{3}$;	$p \sqsupset 32\frac{4}{5}$;	And p has	32
$u=634$;	$p \sqsubset \frac{1}{3}$;	$p \sqsupset 42$;	And p has	41
$u=633$;	$p \sqsubset 1$;	$p \sqsupset 52\frac{1}{5}$;	And p has	51
$u=632$;	$p \sqsubset \frac{2}{3}$;	$p \sqsupset 61\frac{2}{5}$;	And p has	61
$u=631$;	$p \sqsubset \frac{1}{3}$;	$p \sqsupset 70\frac{3}{5}$;	And p has	70
$u=630$;	$p \sqsubset 1$;	$p \sqsupset 80\frac{4}{5}$;	And p has	79 Values.
$u=629$;	$p \sqsubset \frac{2}{3}$;	$p \sqsupset 90$;	And p has	89
$u=628$;	$p \sqsubset \frac{1}{3}$;	$p \sqsupset 99\frac{1}{5}$;	And p has	99
$u=627$;	$p \sqsubset 1$;	$p \sqsupset 109\frac{2}{5}$;	And p has	108
$u=626$;	$p \sqsubset \frac{2}{3}$;	$p \sqsupset 118\frac{3}{5}$;	And p has	118
$u=625$;	$p \sqsubset \frac{1}{3}$;	$p \sqsupset 127\frac{4}{5}$;	And p has	127
$u=624$;	$p \sqsubset 1$;	$p \sqsupset 138$;	And p has	136
$u=623$;	$p \sqsubset \frac{2}{3}$;	$p \sqsupset 147\frac{1}{5}$;	And p has	147
&c.	&c.	&c.		&c.

Whence we may conclude, that when for u, are ſubſtituted the terms of the ſeries 638, 623, 608, 593, &c. decreaſing by $(3 \times 5 =) 15$; then the numbers of the values of p will be the correſponding terms of the ſeries, 4, 147, 290, 433, &c. encreaſing by 143.

And, when u is ſeverally equal to 637, 622, 607, 592, &c. then the numbers of the values of p will be, reſpectively, 13, 156, 299, 442, &c. And ſo on.

Now ſince the arithmetical progreſſions, expreſſing the values of z (when thoſe values are limited by the column A, and the terms of the ſeries 638, 633, 628, 623,

&c. are ſeverally ſubſtituted for u) are 1, 6, 11, 16, &c. 2, 7, 12, 17, &c. And 4, 9, 14, 19, &c. Therefore,

When	Then $p - n$, that is to ſay the number of the terms in which the column A limits the values of z, will be	Therefore		
		Terms of the progreſſion, 1, 6, 11, 16, &c.	Terms of the progreſſion, 2, 7, 12, 17, &c. And	Terms of the progreſſion, 4, 9, 14, 19, &c. will be uſeful.
$u = 638$;	$4 - 2 = 2$;	1	1	0
$u = 633$;	$51 - 26 = 25$;	9	8	8
$u = 628$;	$99 - 52 = 47$;	16	16	15
$u = 623$;	$147 - 77 = 70$;	24	23	23
$u = 618$;	$194 - 102 = 92$;	31	30	30
$u = 613$;	$242 - 127 = 115$;	39	38	38
$u = 608$;	$290 - 153 = 137$;	46	46	45
$u = 603$;	$337 - 177 = 160$;	54	53	53
$u = 598$;	$385 - 202 = 183$;	61	61	61
$u = 593$;	$433 - 228 = 205$;	69	68	68
$u = 588$;	$480 - 252 = 228$;	76	76	76
$u = 583$;	$528 - 278 = 250$;	84	83	83
$u = 578$;	$576 - 303 = 273$;	91	91	91
$u = 573$;	$623 - 328 = 295$;	99	98	98
$u = 568$;	$671 - 353 = 318$;	106	106	106
$u = 563$;	$719 - 378 = 341$;	114	114	113
$u = 558$;	$766 - 403 = 363$;	121	121	121
$u = 553$;	$814 - 428 = 386$;	129	129	128
$u = 548$;	$862 - 454 = 408$;	136	136	136
$u = 543$;	$909 - 478 = 431$;	144	144	143
&c.	&c.	&c.	&c.	&c.

Hence, when the terms of the ſeries 638, 543, 448, 353, &c. decreaſing by $(5 \times 19 =) 95$, are ſeverally ſubſtituted for u, then the two firſt of the arithmetical progreſſions, expreſſing the numbers of the values of z, will ſeverally contain, 1, 144, 287, 430, &c. terms, and the other arithmetical progreſſion will contain, 0, 143, 286, 429, &c. terms.

And when, for *u*, the terms of the ſeries, 633, 538, 443, 348, &c. are ſubſtituted; then the firſt of theſe arithmetical progreſſions will contain, 9, 152, 295, 438, &c. terms; and the remaining two will, each, contain 8, 151, 294, 437, &c. terms; and ſo on.

From what has been ſaid, the number of anſwers which the queſtion is capable of, while, for *u*, are ſeverally ſubſtituted the terms of the ſeries, 638, 633, 628, &c. and the values of *z* are limited by the column A, may be determined either exactly or nearly, by queſt. 13.

We might now proceed to find the ſeries that would reſult from ſubſtituting, 637, 632, 627, 622, &c. ſeverally for *u*; but, as the method above ſhewn, is very eaſy, and a ſufficient number of examples have been given thereto, it is left for the reader's practice.

COROL.

From the whole it may be concluded, that the number of anſwers, in affirmative integers, which an equation, containing never ſo many indetermined quantities, is capable of, may be found, if the labour neceſſary thereto be not a hindrance thereof, by purſuing methods ſimilar to the above: And 'tis hoped, that the reader will excuſe the breaking off this queſtion here, without either pointing out the four remaining claſſes of ſeries, which will reſult while the numbers of the values of *z* are limited by the column A, or giving the ſums of any one ſeries in figures, as the reaſons of ſo doing, are the fear of growing tedious, and the want of an artifice to ſhorten the operations.

SCHOLIUM. Queſtion 12, the firſt of thoſe which contain four indetermined quantities, is of the eaſieſt kind poſſible; the coefficient of z, being the ſum of the coefficients of x and y; and the coefficient of u, being a multiple of all the other coefficients: And the laſt queſtion is of the moſt difficult kind that can happen, requiring the ſummation of $(\overline{3+5}\times 19=)$ 152 ſeries of the ſecond Order in its ſolution: It would be eaſy enough to enumerate all the properties, which the coefficients of the indetermined quantities muſt have, to render the ſolution more difficult than the firſt, and leſs difficult than the laſt; but as the greateſt part of them may be collected from what was ſaid before, concerning equations which have but three indetermined quantities, it will be needleſs to add more than the following, *viz.* that when the coefficient of u is a multiple of the coefficient of z, then the number of thoſe ſeries, whoſe ſecond differences are equal, will be, as few as the number of arithmetical progreſſions would have been, if the equation had but three indetermined quantities.

QUESTION XV.

The value of n terms of the ſeries $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$, &c. is required.

SOLUTION.

If $\frac{z}{r-1}=\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}$, &c. *ad infinitum*,

Then it will appear that $z=1$.

For, if $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}$, &c. be multiplied by $r-1$

$$\begin{array}{l} \hline 1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}, \text{ \&c.} \\ \quad -\frac{1}{r}-\frac{1}{r^2}-\frac{1}{r^3}, \text{ \&c.} \\ \hline \end{array}$$

Product $= 1+0+0+0$, &c.

Therefore $\frac{1}{r-1}$ is the value of the ſeries, if it be infinitely continued.

But ſince the ſum of n terms, only, are required, it will be neceſſary to ſubtract the value of all the terms after the nth term, from the value above found.

Now the terms, which follow the nth term, are $\frac{1}{r^{n+1}}+\frac{1}{r^{n+2}}+\frac{1}{r^{n+3}}$, &c. *ad infinitum*, which $\left(\text{becauſe } \frac{1}{r^{n+1}}=\frac{1}{r^n}\times\frac{1}{r};\ \frac{1}{r^{n+2}}=\frac{1}{r^n}\times\frac{1}{r^2}, \text{ \&c.}\right)$ will become $\frac{1}{r^n}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$, &c. *ad infinitum*.

Now ſince $\frac{1}{r-1}=\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$, &c. as above.

Th. $\frac{1}{r^n}\times\frac{1}{r-1}=\frac{1}{r^n}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$, &c. *ad infinitum*.

Whence the ſum of n terms of the given ſeries will be $\frac{1}{r-1}-\frac{1}{r^n}\times\frac{1}{r-1}$.

If the above ſeries be applied to computations of compound intereſt, that is, if r be the amount of 1 £. in 1 year; then $\frac{1}{r^n}$ will be the preſent worth of 1 £. due at the end of n years. Let $\frac{1}{r^n}$, therefore $=p$,

Then $\frac{1}{r-1}-\frac{1}{r^n}\times\frac{1}{r-1}$ will become $\frac{1}{r-1}-\frac{p}{r-1}$;

Therefore $\frac{1-p}{r-1}$ will be the value required.

SCHOLIUM. The given ſeries, continued to n terms, *viz.* $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}$ (n*), is equal to the preſent worth of an annuity to continue n years certain: Now if ſuch an annuity be denoted by A, then $A=\frac{1-p}{r-1}$.

* *When the Letter* (n) *is placed, as above, after the initial terms of a ſeries, it is deſigned to denote that the ſeries is to be continued to* n *terms, and no more.*

QUESTION XVI.

The value of n terms of the ſeries $\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}$, &c. is required.

SOLU-

SOLUTION.

If $\frac{z}{\overline{r-1}|^{2}} = \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4}$, &c. *ad infinitum.*

Then $z = r$.

For if $\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4}$, &c. be multiplied

By $rr - 2r + 1$

$$\begin{array}{rrrrl} r + & 2 + & \frac{3}{r} + & \frac{4}{rr}, & \&c. \\ & -\;2 - & \frac{4}{r} - & \frac{6}{rr}, & \&c. \\ & & +\;\frac{1}{r} + & \frac{2}{rr}, & \&c. \\ \hline \end{array}$$

The product is $= r + 0 + 0 + 0$, &c.

Therefore $\frac{r}{\overline{r-1}|^{2}}$ is the value of the ſeries infinitely continued.

Now the terms, which follow the *n*th term, in the given ſeries are $\frac{n+1}{r^{n+1}} + \frac{n+2}{r^{n+2}} + \frac{n+3}{r^{n+3}}$, &c. *ad infin.*

Or $\frac{1}{r^n} \times \overline{\frac{n+1}{r} + \frac{n+2}{r^2} + \frac{n+3}{r^3}}$, &c. *ad infin.*

Which ſeries may be divided into two others, *viz.*

$\frac{1}{r^n}\times\overline{\frac{n}{r}+\frac{n}{r^2}+\frac{n}{r^3}}$, &c. $=\frac{n}{r^n}\times\frac{1}{r-1}$ by queſt. 15; And

$\frac{1}{r^n}\times\overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}}$, &c. $=\frac{1}{r^n}\times\frac{r}{\overline{r-1}^2}$, by the above.

Now if the ſum of theſe two ſeries be ſubtracted from the former, the remainder will be the value required, *viz.* $\frac{r}{\overline{r-1}^2}-\frac{n}{r^n}\times\frac{1}{r-1}-\frac{1}{r^n}\times\frac{}{\overline{r-1}^2}$; in which if $\frac{1}{r^n}=p$, it will be $\frac{r}{\overline{r-1}^2}-\frac{rp}{\overline{r-1}^2}-\frac{np}{r-1}$;

Or $\frac{\overline{1-p}\times r}{\overline{r-1}^2}-\frac{np}{r-1}$.

QUESTION XVII.

The value of *n* terms of the ſeries, $\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}$, &c. is required?

SOLUTION.

If $\frac{z}{\overline{r-1}^3}=\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}$, &c.

Then $z=(rr+r=)\,\overline{r+1}\times r$.

For if $\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}$, &c. be multiplied

By r^3-3r^2+3r-1

$$rr+4r+9+\frac{16}{r}+\frac{25}{rr}, \&c.$$

$$-3r-12-\frac{27}{r}-\frac{48}{rr}, \&c.$$

$$+3+\frac{12}{r}+\frac{27}{rr}, \&c.$$

$$-\frac{1}{r}-\frac{4}{rr}, \&c.$$

The product is $\left.\right\}$ $rr+r+0+0+0$, &c.

Therefore the given ſeries infinitely continued is worth $\frac{\overline{r+1}\times r}{\overline{r-1}^3}$.

Now the terms, which follow the nth term, in the given ſeries are $\frac{\overline{n+1}^2}{r^{n+1}}+\frac{\overline{n+2}^2}{r^{n+2}}+\frac{\overline{n+3}^2}{r^{n+2}}$, &c.

Or $\frac{nn+2n+1}{r^{n+1}}+\frac{nn+4n+4}{r^{n+2}}+\frac{nn+6n+9}{r^{n+2}}$, &c.

Which ſeries may be divided into three others, *viz.*

$\frac{nn}{r^n}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}, \&c.}=\frac{nn}{r^n}\times\frac{1}{r-1}$ By queſt. 15.

$\frac{2n}{r^n}\times\overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}, \&c.}=\frac{2n}{r^n}\times\frac{r}{\overline{r-1}^2}$ By queſt. 16.

$\frac{1}{r^n}\times\overline{\frac{1}{r}\quad\frac{4}{r^2}+\frac{9}{r^3}, \&c.}=\frac{1}{r^n}\times\frac{\overline{r+1}\cdot r}{\overline{r-1}^3}$ By above.

In which three ſeries, writing p for $\frac{1}{r^n}$, and ſubtracting their ſum from the above found value of the whole ſeries, there remains $\frac{1-p\times\overline{r+1}\times r}{\overline{r-1}^3} - \frac{2nrp}{\overline{r-1}^2} - \frac{nnp}{r-1}$ the value of n terms thereof, which was required.

QUESTION XVIII.

The value of n terms of the ſeries, $\frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3} + \frac{64}{r^4} + \frac{125}{r^5}$, &c. is required?

SOLUTION.

If $\frac{z}{\overline{r-1}^4} = \frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3} + \frac{64}{r^4} + \frac{125}{r^5}$, &c. *ad infin.*

Then $z = r^3 + 4r^2 + r$, as will appear, if the multiplication be performed, after the manner of the three laſt queſtions.

But $\overline{r+2}^2 = rr + 4r + 4$,

And $\overline{r+2}^2 \times r = r^3 + 4rr + 4r$;

Th. $\overline{r+2}^2 \times r - z = 3r$,

And $z = \overline{r+2}^2 \times r - 3r$.

Therefore the ſeries, infinitely continued, will be in value $\frac{r\cdot\overline{r+2}^2 - 3r}{\overline{r-1}^4}$.

Now the terms, which follow the *n*th term, will be

$$\frac{\overline{n+1}^3}{r^{n+1}} + \frac{\overline{n+2}^3}{r^{n+2}} + \frac{\overline{n+3}^3}{r^{n+3}}, \text{ \&c.}$$

Or $\frac{n^3+3n^2+3n+1}{r^{n+1}} + \frac{n^3+6n^2+12n+8}{r^{n+2}} +$

$$\frac{n^3+9n^2+27n+27}{r^{n+3}}, \text{ \&c.}$$

Which ſeries may be divided into 4 others, *viz.*

$\frac{n^3}{r^n} \times \overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}, \text{ \&c.}} = \frac{n^3}{r^n} \times \frac{1}{r-1}$ By queſt. 15.

$\frac{3n^2}{r^n} \times \overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}, \text{ \&c.}} = \frac{3n^2}{r^n} \times \frac{r}{\overline{r-1}^2}$ By queſt. 16.

$\frac{3n}{r^n} \times \overline{\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}}, \text{ \&c.} = \frac{3n}{r^n} \times \frac{r\cdot\overline{r+1}}{\overline{r-1}^2}$ By queſt. 17.

$\frac{1}{r^n} \times \overline{\frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}}, \text{ \&c.} = \frac{1}{r^n} \times \frac{r\cdot\overline{r+2}^2-3r}{\overline{r-1}^4}$, as above.

Whence if p be written for $\frac{1}{r^n}$ in the value of thoſe 4 ſeries, and their ſum be ſubtracted from the before found value of the given ſeries, infinitely continued, the remainder will be,

$$\frac{\overline{1-p}\times\overline{r\cdot\overline{1+2}^2-3r}}{\overline{r-1}^4} - \frac{\overline{r+1}\times 3nrp}{\overline{r-1}^3} - \frac{3nnrp}{\overline{r-1}^2} - \frac{n^3p}{r-1},$$

which is the value of n terms as was required.

QUES-

QUESTION XIX.

The value of n terms of the ſeries $\frac{1}{r}+\frac{16}{rr}+\frac{81}{r^3}+\frac{256}{r^4}$ $+\frac{625}{r^5}$, is required?

SOLUTION.

If $\frac{z}{\overline{r-1}^{5}}=\frac{1}{r}+\frac{16}{rr}+\frac{81}{r^3}+\frac{256}{r^4}+\frac{625}{r^5}$, *&c. ad infini*

Then $z=r^4+11r^3+11r^2+r$, as will appear by multiplying both ſides of the equation by $\overline{r-1}^{5}$.

Therefore the ſeries infinitely continued will be $=\frac{r^4+11r^3+11r^2+r}{\overline{r-1}^{5}}$.

Now the terms, which follow the nth term, will be

$$\frac{\overline{n+1}^{4}}{r^{n+1}}+\frac{\overline{n+2}^{4}}{r^{n+2}}+\frac{\overline{n+3}^{4}}{r^{n+3}}, \&c.$$

Or

$$\frac{n^4+4n^3+6n^2+4n+1}{r^{n+1}}$$

$$+\frac{n^4+8n^3+24n^2+32n+16}{r^{n+2}}$$

$$+\frac{n^4+12n^3+54n^2+108n+81}{r^{n+3}}$$

Which ſeries may be divided into the five following, *viz.*

$$\frac{n^4}{r^n} \times \overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}, \&c. = \frac{n^4}{r^n} \times \frac{1}{r-1} \text{ By queſt. 15.}$$

$$\frac{4n^3}{r^n} \times \overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}}, \&c. = \frac{4n^3}{r^n} \times \frac{r}{\overline{r-1}^2} \text{ By queſt. 16.}$$

$$\frac{6n^2}{r^n} \times \overline{\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}}, \&c. = \frac{6n^2}{r^n} \times \frac{\overline{r+1} \times r}{\overline{r-1}^3} \text{ By queſt. 17.}$$

$$\frac{4n}{r^n} \times \overline{\frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}}, \&c. = \frac{4n}{r^n} \times \frac{r^3+4r^2+r}{\overline{r-1}^4} \quad 18.$$

$$\frac{1}{r^n} \times \overline{\frac{1}{r}+\frac{16}{rr}+\frac{81}{r^3}}, \&c. = \frac{1}{r^n} \times \frac{r^4+11r^3+11r^2+r}{\overline{r-1}^5}$$

Whence, if we write p for $\frac{1}{r^n}$, and ſubtract theſe 5 ſeries from the value of the given ſeries, infinitely coninued, there will remain

$$\overline{1-p} \times \frac{r^4+11r^3+11r^2+r}{\overline{r-1}^5} - 4np \times \frac{r^3+4r^2+r}{\overline{r-1}^4}$$

$$\Bigl(-6n^2p \times \frac{rr+r}{\overline{r-1}^3} - 4n^3p \times \frac{r}{\overline{r-1}^2} - n^4p \times \frac{1}{r-1}.$$

the value of n terms of the required ſeries.

Corol. If we put $\frac{1}{r-1} = P$; $\frac{r}{\overline{r-1}^2} = Q$; $\frac{rr+r}{\overline{r-1}^3} = R$; $\frac{r^3+4r^2+r}{\overline{r-1}^4} = S$; $\frac{r^4+11r^3+11r^2+r}{\overline{r-1}^5}$ T, &c.

then may the ſum of n terms of the ſeries $\frac{1^m}{r} + \frac{2^m}{r^2}$

+

$+\frac{3^m}{r^3}+\frac{4^m}{r^4}+\frac{5^m}{r^5}$, &c. be expreſſed by $-n^m p\times P$ $-mn^{m-1}p\times \mathcal{Q}-\frac{m\cdot\overline{m-1}}{1\cdot 2}n^{m-2}p\times R-$ $\frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{1\cdot 2\cdot 3}n^{m-3}p\times S$, &c. $+\overline{1-p}\times Z$.

Where Z ſignifies the ſum of the ſeries $\frac{1^m}{r}+\frac{2^m}{r^2}+\frac{3^m}{r^3}$, &c. *ad infinitum.*

It only remains, therefore, in the ſucceeding queſtions to find the value of *V*, *W*, *X*, *Y*, &c. the ſums of the following ſeries when infinitely continued.

QUESTION XX.

What is *V* the value of $\frac{1}{r}+\frac{32}{rr}+\frac{243}{r^3}+\frac{1024}{r^4}$, &c. *ad infinitum.*

SOLUTION.

Now to proceed as before,

$$\begin{array}{lrrrrrl}
\text{Multiply} & \frac{1}{r} & +\frac{32}{rr} & +\frac{243}{r^3} & +\frac{1024}{r^4} & +\frac{3125}{r^5}, & \text{\&c.} \\
\text{by} & r^6 & -6r^5 & +15r^4 & -20r^3 & +15r^2, & \text{\&c.} \\
\hline
 & r^5 & +32r^4 & +243r^3 & +1024r^2 & +3125r, & \text{\&c.} \\
 & & -6r^4 & -192r^3 & -1458r^2 & -6144r, & \text{\&c.} \\
 & & & +15r^3 & +480r^2 & +3645r, & \text{\&c.} \\
 & & & & -20r^2 & -640r, & \text{\&c.} \\
 & & & & & +15r, & \text{\&c.} \\
\hline
\text{The prod. will be} & r^5 & +26r^4 & +66r^3 & +26r^2 & +r, & \text{\&c.}
\end{array}$$

$$\text{Therefore } V = \frac{r^5+26r^4+66r^3+26r^2+r}{\overline{r-1}|^6}$$

COROL.

From the beforegoing ſolutions, we may conclude,

1*ſt*. That the number of terms, in the numerator of the fraction, which exhibits the value of theſe kind of infinite ſeries, will be always equal to *m*, the index of that power to which the numbers, 1, 2, 3, *&c.* are raiſed in the numerators of the terms conſtituting the ſeries.

2*d*. That, after the numeral coefficients of half thoſe terms are found, the following coefficients will be the ſame with the former, but in a reverſed order; after the manner of the unciæ of a binomial.

3*d*. That the denominator of the ſaid fraction will always be $r-1$ raiſed to a power $(m+1)$ whoſe index is greater, by unity, than the index of that power to which the numbers, 1, 2, 3, *&c.* are raiſed in the numerators of the terms conſtituting the ſeries.

Alſo, from the nature of the operation by which the numerator of the ſaid fraction is found, it will appear:

4. That the coefficient of the firſt term thereof will be unity.

5. The coefficient of the ſecond term, $2^m - \frac{m+1}{1}$.

6. That of the third, $3^m - 2^m \times \frac{m+1}{1} + \frac{\overline{m+1}\cdot m}{1\cdot 2}$.

7. That of the fourth, $4^m - 3^m \times \frac{m+1}{1} + 2^m \times \frac{\overline{m+1}\cdot m}{1\cdot 2} - \frac{\overline{m+1}\cdot m\cdot \overline{m-1}}{1\cdot 2\cdot 3}$.

And therefore Z, the value of the ſeries $\frac{1^m}{r} + \frac{2^m}{r^2} + \frac{3^m}{r^3} + \frac{4^m}{r^4}$, *&c. ad infinitum*, will be

$$\frac{r}{\overline{r-1}\,|^{m+1}} \times \left\{ \begin{array}{l} r^m + r^{m-1} \times \overline{2^m - \frac{m+1}{1}} + \\ r^{m-2} \times \overline{3^m - 2^m \times \frac{m+1}{1} + \frac{\overline{m+1}\times m}{1\times 2}} + \\ r^{m-3} \left\{ \begin{array}{l} 4^m - 3^m \times \frac{m+1}{1} + 2^m \times \frac{\overline{m+1}\times m}{1\cdot 2} \\ - \frac{\overline{m+1}\times m \times \overline{m-1}}{1\times 2\times 3} \end{array} \right\} (m) \end{array} \right.$$

Thus are we furniſhed with the means of finding the numbers *W*, *X*, *Y*, *&c.*

Firſt, for *W*; Where $m=6$: Then $2^6=64$; $\frac{m+1}{1}=7$; And $64-7=57$: $3^6=729$; $64\times 7=448$; $7\times\frac{6}{2}=21$; And $729-448+21=302$:

Th. $W = \dfrac{r^6 + 57r^5 + 302r^4 + 302r^3 + 57r^2 + r}{\overline{r-1}^{7}}$.

Secondly, to find X, $m=7$: Then $2^7=128$; $m+1=8$; And $128-8=120$: $3^7=2187$; $128\times8=1024$; $8\times\frac{7}{2}=28$, And $2187-1024+28=1191$: $4^7=16384$; $2187\times8=17496$; $128\times28=3584$; $28\times\frac{6}{3}=56$; And $16384-17496+3584-56=2416$.

Th. X is

$$= \frac{r^7 + 120r^6 + 1191r^5 + 2416r^4 + 1191r^3 + 120r^2 + r}{\overline{r-1}^{8}}.$$

Thirdly, to find Y, $m=8$: Then $2^8=256$; $m+1=9$; And $256-9=247$: $3^8=6561$; $256\times9=2304$; $9\times\frac{8}{2}=36$; And $6561-2304+36=4293$; $4^8=65536$; $6561\times9=59049+9216-84=15619$.

Th. $Y=$

$$\frac{r^8 + 247r^7 + 4293r^6 + 15619r^5 + 15619r^4 + 4243r^3 \&c.}{\overline{r-1}^{9}}.$$

As the numeral values of theſe expreſſions, at the different rates of intereſt, and their logarithms, will be of great ſervice in the ſpeedy reſolution of queſtions, relating to the values of combined lives, the following table is inſerted.

Rate

Rate per Ct.	P	Q	R	S	T	V
3	$\frac{100}{3}$	$\frac{10300}{9}$	77440,741	7859663	1063596960	179912043600
3½	$\frac{200}{7}$	$\frac{41400}{49}$	49124,782	4283960	498114240	72397376300
4	25	650	33150	2535650	258603154	32967678466
4½	$\frac{200}{9}$	$\frac{41800}{81}$	23451,575	1598358	145249496	16499293600
5	20	420	17220	1058820	86806019	8895852030
6	$\frac{50}{3}$	$\frac{2650}{9}$	10109,259	520480	35729490	3065913930

	Log. P	Log. Q	Log. R	Log. S	Log. T	Log. V
3	1,5228787	3,0585946	4,8889693	6,8954039	9,0267769	11,2550600
3½	1,4559320	2,9268043	4,6913007	6,6318456	8,6973292	10,8597228
4	1,3979400	2,8129134	4,5204835	6,4040893	8,4126338	10,5180883
4½	1,3467875	2,7126913	4,3701721	6,2036741	8,1621147	10,2174655
5	1,3010300	2,6232493	4,2360334	6,0248221	7,9385499	9,9491876
6	1,2218487	2,4690033	4,0047192	5,7164035	7,5530265	9,4865597

QUESTION XXI.

If S be the ſum of n terms of the arithmetical progreſſion a, $a-d$, $a-2d$, $a-3d$, &c. whoſe common difference d is known; and if Z be the ſum of n terms of another arithmetical progreſſion b, $b-\delta$, $b-2\delta$, $b-3\delta$, &c. whoſe common difference δ is alſo known; and if there be farther given SZ the product of the ſaid ſums, it is thence required to find the ſum of n terms of the following ſeries, *viz.* $ab+\overline{a-d}\times\overline{b-\delta}+\overline{a-2d}\times\overline{b-2\delta}$, &c. which is formed by multiplying the ſeveral correſponding terms of each arithmetical progreſſion together?

SOLUTION.

By queſt. 3. part 2. vol. I. $S=na-\dfrac{n\cdot\overline{n-1}}{1\cdot2}d.$

And $Z=nb-\dfrac{n\cdot\overline{n-1}}{1\cdot2}\delta;$

Th. $SZ=nnab-\dfrac{nn\cdot\overline{n-1}}{1\cdot2}bd-\dfrac{nn\cdot\overline{n-1}}{1\cdot2}a\delta+\dfrac{nn\cdot\overline{n-1}^2}{2\cdot2}d\delta.$ which expreſſes the given product.

The terms of the ſeries $ab+\overline{a-d}\times\overline{b-\delta}+\overline{a-2d}\times\overline{b-2\delta}$, &c. may be repreſented as follows, *viz.*

$$\begin{array}{lllll}
ab & =ab, & & & \\
\overline{a-d}\times\overline{b-\delta} & =ab- & bd- & a\delta+ & d\delta, \\
\overline{a-2d}\times\overline{b-2\delta} & =ab- & 2bd- & 2a\delta+ & 4d\delta, \\
\overline{a-3d}\times\overline{b-3\delta} & =ab- & 3bd- & 3a\delta+ & 9d\delta; \\
\&c. & \&c. & \&c. & \&c. & \&c.
\end{array}$$

The ſum of which may be divided into the four following ſeries, *viz.*

$$ab\times\overline{1+1+1+1\ (n)}=nab,$$

$$\left.\begin{aligned} bd\times\overline{0+1+2+3\ (n)}&=\frac{n\cdot\overline{n-1}}{1\cdot2}bd,\ \text{See qu. 1.}\\ a\delta\times\overline{0+1+2+3\ (n)}&=\frac{n\cdot\overline{n-1}}{1\cdot2}a\delta,\quad 1\\ d\delta\times\overline{0+1+4+9\ (n)}&=\frac{n\cdot\overline{n-1}\cdot\overline{2n-1}}{1\cdot2\cdot3}d\delta;\ 33\end{aligned}\right\}\text{Part 2. Vol. I.}$$

Therefore putting $\Sigma=$ the ſum of n terms of the ſaid ſeries ab, $\overline{a-d}\times\overline{b-\delta}+\overline{a-2d}\times\overline{b-2\delta}$, &c.

$$\Sigma=nab-\frac{n\cdot\overline{n-1}}{1\cdot2}bd-\frac{n\cdot\overline{n-1}}{1\cdot2}a\delta+\frac{n\cdot\overline{n-1}\cdot\overline{2n-1}}{1\cdot2\cdot3}d\delta;$$

Now if the above found value of SZ be divided by n

$$\frac{SZ}{n}=nab-\frac{n\cdot\overline{n-1}}{1\cdot2}bd-\frac{n\cdot\overline{n-1}}{1\cdot2}a\delta+\frac{n\cdot\overline{n-1}\cdot\overline{n-1}}{1\cdot2\cdot2}d\delta;$$

Whence by ſubtraction,

$$\Sigma-\frac{SZ}{n}=\frac{n\cdot\overline{n-1}\cdot\overline{2n-1}}{1\cdot2\cdot3}d\delta-\frac{n\cdot\overline{n-1}\cdot\overline{n-1}}{1\cdot2\cdot2}d\delta,\ \text{Or}$$

$$\Sigma-\frac{SZ}{n}=\frac{n\cdot\overline{n-1}}{1\cdot2}d\delta\times\overline{\frac{2n-1}{3}-\frac{n-1}{2}};$$

$$\text{But}\ \frac{n+1}{6}=\frac{2n-1}{3}-\frac{n-1}{2};\ \text{Therefore}$$

$$\Sigma-\frac{SZ}{n}=\left(\frac{n\cdot\overline{n-1}}{1\cdot2}d\delta\times\frac{n+1}{6}=\right)\frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}d\delta.$$

Th. $\Sigma=\dfrac{SZ}{n}+\dfrac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}d\delta$ is the value of n terms of the required ſeries.

QUESTION XXII.

$$\begin{aligned} \text{If } S &= a + \overline{a-d} + \overline{a-2d}\ (n), \\ Z &= b + \overline{b-\delta} + \overline{b-2\delta}\ (n), \\ W &= c + \overline{c-\Delta} + \overline{c-2\Delta}\ (n), \\ \text{And } \Sigma &= abc + \overline{a-d} \times \overline{b-\delta} \times \overline{c-\Delta} + \\ &\quad (\overline{a-2d} \times \overline{b-2\delta} \times \overline{c-2\Delta}\ (n); \end{aligned}$$

Then having given SZW, (the product of the three ſums of the different arithmetical progreſſions) and what elſe may be neceſſary given, Σ is required?

SOLUTION.

$$\text{Since } S = na - \frac{n \cdot \overline{n-1}}{1 \cdot 2} d,$$

$$Z = nb - \frac{n \cdot \overline{n-1}}{1 \cdot 2} \delta,$$

$$\text{And } W = nc - \frac{n \cdot \overline{n-1}}{1 \cdot 2} \Delta;$$

It will follow, that,

$$SZW = \left\{ \begin{array}{l} n^3\, abc - \dfrac{n^3 \cdot \overline{n-1}}{2} + \overline{ab\Delta + ac\delta + bcd} + \\ \dfrac{n^3 \cdot \overline{n-1}^2}{2 \cdot 2} \times \overline{a\delta\Delta + bd\Delta + ca\delta} - \dfrac{n^3 \cdot \overline{n-1}^3}{2 \cdot 4} d\delta\Delta \end{array} \right.$$

As will appear by the continual multiplication of the ſeveral values of S, Z and W; Therefore

$$\frac{SZW}{nn}=\left\{\begin{array}{l} nabc-\frac{n\cdot\overline{n-1}}{1\cdot2}\times\overline{ab\Delta+ac\delta+bcd}+ \\ \frac{n\cdot\overline{n-1}|^2}{2\cdot2}\times\overline{a\delta\Delta+bd\Delta+cd\delta}-\frac{n\cdot\overline{n-1}|^3}{2\cdot4}d\delta\Delta; \end{array}\right.$$

And, by proceeding as in the laſt queſtion, it will appear that n terms of the ſeries $abc+\overline{a-d}\times\overline{b-\delta}\times\overline{c-\Delta}+\overline{a-2d}\times\overline{b-2\delta}\times\overline{c-2\Delta}+\overline{a-3d}\times\overline{b-3\delta}\times\overline{c-3\Delta}$, *&c.* That is

$$\Sigma=\left\{\begin{array}{l} nabc-\frac{n\cdot\overline{n-1}}{1\cdot2}\times\overline{ab\Delta+ac\delta+bcd}+ \\ \frac{n\cdot\overline{n-1}\cdot\overline{2n-1}}{1\cdot2\cdot3}\times\overline{a\delta\Delta+bd\Delta+cd\delta}-\frac{n^2\cdot\overline{n-1}|^2}{2\cdot2}d\delta\Delta; \end{array}\right.$$

$$\text{Th}\cdot\Sigma-\frac{SZW}{nn}=\left\{\begin{array}{l} \overline{\frac{2n-1}{3}-\frac{n-1}{2}}\times\frac{n\cdot\overline{n-1}}{1\cdot2}\times\overline{a\delta\Delta+bd\Delta+cd\delta} \\ -\overline{\frac{n\cdot\overline{n-1}}{2}-\frac{\overline{n-1}|^2}{4}}\times\frac{n\cdot\overline{n-1}}{1\cdot2}d\delta\Delta: \end{array}\right.$$

Now $\frac{n+1}{6}=\frac{2n-1}{3}-\frac{n-1}{2}$,

And $\frac{\overline{n+1}\cdot\overline{n-1}}{2\cdot2}=\frac{n\cdot\overline{n-1}}{1\cdot2}-\frac{\overline{n-1}|^2}{4}$;

$$\text{Th}.\Sigma-\frac{SZW}{nn}=\left\{\begin{array}{l} \frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}\times\overline{a\delta\Delta+bd\Delta+cd\delta} \\ -\frac{\overline{n+1}\cdot n\cdot\overline{n-1}|^2}{2\cdot2.2}d\delta\Delta, \end{array}\right.$$

$$\text{And}\ \Sigma=\frac{SZW}{nn}+\left\{\begin{array}{l} \frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}\times\overline{a\delta\Delta+bd\Delta+cd\delta} \\ -\frac{\overline{n+1}\cdot n\cdot\overline{n-1}|^2}{2\cdot2\cdot2}d\delta\Delta. \end{array}\right.$$

QUESTION XXIII.

Suppoſe a round piece of metal, equably formed, having two oppoſite faces, one white, the other black, be thrown up, in order to ſee which of thoſe faces will be uppermoſt, after the metal is fallen to the ground; when, if the white face appears uppermoſt, a perſon is to be entitled to 5 *l.* or any other ſum of money: It is required to determine, before the event, what chance, or probability, that perſon hath of receiving the 5 *l.* and what ſum he may reaſonably expect ſhould be paid to him, in conſideration of his reſigning his chance to another?

SOLUTION.

Since, by ſuppoſition, there is nothing in the form of the metal that can incline it to ſhew one face rather than the other; and ſince it muſt neceſſarily ſhew, either the white, or the black face, it will follow, that there is an equal chance for the appearance of either face; or, in other words, that there is one chance, out of two, for the appearance of the white face; and, conſequently, the probability thereof may be expreſs'd by the fraction $\frac{1}{2}$: If, therefore, any other perſon ſhould be willing to purchaſe this chance, the proprietor may reaſonably expect $\frac{1}{2}$ of the 5 *l.* in conſideration of his reſigning thereof.

QUESTION XXIV.

Suppoſe there are three cards, each of different ſuits, *viz*, one heart, one diamond, and one club, laid on a table with their faces downward; out of which, if a perſon at one trial takes the heart, he is to be entitled to five pounds, or any other ſum of money: It is required to determine, before the event, what chance, or probability, he hath of winning and miſſing the ſaid five pounds, and what ſum he may reaſonably expect to be paid to him, in conſideration of his reſigning the chance to another?

SOLUTION.

Since there is nothing on the outſide, whereby the perſon chooſing can judge or determine whether of the three cards, expoſed to his view, is the heart; and ſince he is to have but one choice, it will follow, that he hath but one chance in three, for obtaining the five pounds, and that the probability thereof may be expreſſed by the fraction $\frac{1}{3}$: Again, ſince there will be two cards remaining after he has made his choice, either of which may be the heart; therefore there are two chances out of three, that he will miſs it, and the probability thereof may be expreſſed by the fraction $\frac{2}{3}$: Laſtly, he may reaſonably expect $\frac{1}{3}$ of the five pounds, as a conſideration for transferring his chance to another.

QUESTION XXV.

Suppoſe that there are five counters, four whereof are black, and one white; out of which (being mixed together) a perſon blindfolded is to draw one, and is to be entitled to five pounds, or any other ſum, if he happens to draw that which is white: It is required to determine, before the event, what chance, or probability, he has of winning and miſſing the ſaid five pounds, and what ſum he may reaſonably expect to be paid to him, for reſigning his chance to another.

SOLUTION.

Since the perſon, who is to draw the counter, is ſuppoſed to be deprived of his ſight, he cannot form any judgment, which of the five counters is the white one, or prize, and ſince he is confined to the taking, only, one of them, it will follow, that he hath only one chance in five, for obtaining the five pounds, and that the probability thereof may be expreſſed by the fraction $\frac{1}{5}$: Again, ſince there will be four counters remaining, after he hath drawn out one, either of which may be the prize; therefore there are four chances out of the five for his miſſing it, and the probability thereof may be expreſſed by the fraction $\frac{4}{5}$: Laſtly he will be entitled to $\frac{1}{5}$ of the five pounds, if he transfers his chance to another.

QUESTION XXVI.

Suppoſe there are five counters, three whereof are black, and the other two white, out of which, when mixed together, a perſon blindfolded is to draw one, and is to be entitled to five pounds, or any other ſum, if he happens to draw either of the white ones: It is required to determine, before the event, what chance, or probability, he has of winning and miſſing the ſaid five pounds, and what ſum he may reaſonably expect to be paid to him, for reſigning his chance to another?

SOLUTION.

As the perſon that draws cannot know which three of the five counters are black, and which two are white; and as he is to take but one of them; it is plain that he hath only two out of the five chances for taking a white counter, and the other three for taking a black one; and conſequently the probability of winning may be expreſſed by the fraction $\frac{2}{5}$, and that of miſſing by $\frac{3}{5}$: Laſtly, he ought to receive $\frac{2}{5}$ of the five pounds, if he parts with his chance to another.

SCHOLIUM. What has been ſaid, in theſe four queſtions, concerning cards, counters, *&c.* may be very well conceived to extend to any other things, which are the objects of chance: For inſtance, if at the concluſion of the drawing of a Lottery, there ſhould remain in one wheel five tickets or numbers only, and in the other wheel two equal prizes, and three blanks: Then the poſſeſſor of one of thoſe tickets, would be exactly in the ſtate of the perſon mentioned in the laſt queſtion.

COROLLARIES.

Since, when the number of blanks is	And the number of prizes	And, conſequently, the number of tickets	The probability of having a prize with one ticket is	And the probability of a blank	The ſum of which two probabilities is unity.
1	1		$\frac{1}{2}$	$\frac{1}{2}$	
2	1	3	$\frac{1}{3}$	$\frac{2}{3}$	
	1	5	$\frac{1}{5}$	$\frac{4}{5}$	
3	2	5	$\frac{2}{5}$	$\frac{3}{5}$	

Therefore, *firſt*, when in any lottery, the number of blanks is m, and the number of prizes n, then the probability of having one prize, with one ticket, will be $\frac{n}{m+n}$; and the probability of having a blank $\frac{m}{m+n}$: Or, in other words, *the probability of the happening of any event, reſulting from chance, may be expreſſed by a fraction; whoſe numerator is the number of chances for the events taking place; and the denominator the number of all the chances, whereby it may both happen and fail.* And *the probability of ſuch an event's failing may be expreſſed by a fraction, whoſe numerator is the number of chances for the not happening thereof, and the denominator the ſame with that of the former fraction.*

Secondly, Since the ſum of the two fractions, repreſenting the probabilities of the happening and failing of an event, is unity; therefore, the one of them being given, the other may be found by ſubtraction.

Thirdly, The expectation (that is, the ſum which the perſon, who has a chance for the happening of an event) is entitled to, if he ſurrenders that chance to another, is always the product of the fraction, repreſenting the probability, multiplied into the ſum expected; and if the ſum expected be denoted by unity, then the expectation will be denoted by the probability itſelf. Examples of which will frequently occur hereafter.

QUESTION XXVII.

A perſon, playing with a ſingle die, offers to lay a wager, that he will throw an ace, each time, for two ſucceſſive throws: What probability has he of ſucceeding therein?

SOLUTION.

Suppoſe the wager to be 36*l.* and that, on throwing, the firſt time, an ace did come up: Then becauſe there are ſix faces on the die, one of which (only) is favourable to him, his expectation on the ſecond throw will be ($\frac{1}{6}$ of 36*l.* or) 6*l.* Now we may conceive the firſt event, *viz.* the throwing an ace the firſt throw, as the condition of obtaining this 6*l.* but the probability of this event being alſo $\frac{1}{6}$, the expectation, before the firſt throw, muſt neceſſarily be ($\frac{1}{6}$ of 6*l.* or) 1*l.* which being the $\left(\frac{1}{6}\times\frac{1}{6}=\right)$ $\frac{1}{36}$ part of the ſum wagered, if that wager had been 1*l.* the expectation, and, conſequently, the probability of winning thereof, would neceſſarily have been $\frac{1}{36}l.$ And therefore, the probability of his loſing the wager will be $\left(1-\frac{1}{36}=\right)\frac{35}{36}$; whence it appears, that no perſon ſhould lay that wager, unleſs 35 to 1 in value, be laid againſt him.

QUESTION XXVIII.

A perſon offers to lay a wager of 1 *l.* that out of a purſe containing $\overline{m+n}$ counters, whereof m are black and n white, he will, blindfolded, at the firſt trial, draw a white counter; and alſo that, out of another purſe, containing $\overline{M+N}$ counters, whereof M are black, and N white, he will alſo, blindfolded, at the firſt trial, draw a white counter; and that, if he fails in either trial, his wager ſhall be loſt: It is required to determine the probability he has of ſucceeding therein.

SOLUTION.

Now if, as in the laſt queſtion, we for the preſent ſuppoſe he has already ſucceeded in the firſt trial, then it will follow, from what is beforegoing, that his expectation on the ſecond will be $\frac{N}{M+N}$: And if, as in the ſame queſtion, we conceive the ſucceſs of the firſt trial as the condition of obtaining this expectation, then the probability of ſo doing will be $\frac{n}{m+n}$; which multiplied into that expectation $\left(viz.\ \frac{N}{M+N} \right)$ will give $\frac{n}{m+n} \times \frac{N}{M+N}$ for the probability required.

Therefore, the probability of the happening of two *independent* events, will be equal to the product of the probabilities of their happening ſeparately.

COROL.

COROL.

If the two events are of the ſame kind, then the probability will be $\frac{nn}{\overline{m+n}^2}$

QUESTION XXIX.

A perſon offers to lay a wager of 1 *l.* that out of a purſe containing $m+n$ counters, whereof m are black and n white, he will, blindfolded, at the firſt trial, draw a black counter; and alſo that, out of another purſe containing $M+N$ counters, whereof M are black and N white, he will alſo, blindfolded, draw, at the firſt trial, a white counter; and that if either experiment fails he will loſe his wager: It is required to determine the probability he has of ſucceeding therein?

SOLUTION.

If the argument of the laſt queſtion be followed the anſwer will be $\frac{m}{m+n} \times \frac{N}{M+N}$; but as this queſtion may be taken in a different light from the former, *viz.* that he is to fail of drawing a white counter at the firſt trial, and to ſucceed therein in the ſecond; and then the probability of the failing of the firſt trial $\left(1-\frac{n}{m+n}\right)$ being multiplied by $\left(\frac{N}{M+N}\right)$ the probability of ſucceeding in the ſecond; the product $\left(viz.\ \overline{1-\frac{n}{m+n}} \times \frac{N}{M+N}\right)$ which will be the probability required.

And in like manner, the probability of failing in both experiments will be the product of the probabilities of their ſeparately failing, *viz.* $\overline{1-\frac{n}{m+n}}\times\overline{1-\frac{N}{M+N}}$.

COROL.

If the events, above ſpoken of, be exactly of the ſame kind, then the probability of failing in the firſt trial, and ſucceeding in the ſecond, will be $\overline{1-\frac{n}{m+n}}\times\frac{n}{m+n}$.

And the probability of failing in both $\overline{1-\frac{n}{m+n}}\times\overline{1-\frac{n}{m+n}}$.

QUESTION XXX.

What is the probability of throwing with a ſingle die, one ace, or more, in two throws; that is, either at the firſt or ſecond throw, or at both?

SOLUTION.

Let 6 (the number of the faces of the die) $=n$

Then, the probability of throwing the required face, the firſt throw, will be $\frac{1}{n}$; and the probability of miſſing it $\left(1-\frac{1}{n}=\right)\frac{n-1}{n}$.

Suppoſe it miſſed at the firſt throw; and then the probability of throwing it, the ſecond time, will be alſo $\frac{1}{n}$; but if this probability be valued before the firſt throw is made, it muſt be connected with $\frac{n-1}{n}$ the probability of miſſing it then, (for if it be thrown at firſt, the condition of the queſtion is performed, and there is no occaſion to throw a ſecond time): And the probability of miſſing it the firſt time, and throwing it the ſecond, will be $\frac{n-1}{n} \times \frac{1}{n} = \frac{n-1}{nn}$.

Therefore the probability of throwing the required face, either at the firſt or ſecond throw, will be $\left(\frac{1}{n} + \frac{n-1}{nn} = \frac{n+\overline{n-1}}{nn} = \frac{2n-1}{nn} = \right)\frac{nn-\overline{n-1}^2}{nn}$; In this caſe $\left(\frac{36-25}{36} = \right)\frac{11}{36}$.

This queſtion may be more readily anſwered by finding the probability of miſſing an ace twice, and ſubtracting that probability from unity; for then the remainder will be the probability of throwing one ace at leaſt in two throws.

Thus, the probability of miſſing an ace the firſt throw is $\frac{n-1}{n}$, and the probability of miſſing it the ſecond throw is alſo $\frac{n-1}{n}$; therefore the probability of miſſing an ace twice is $\left(\frac{n-1}{n} \times \frac{n-1}{n} = \right)\frac{\overline{n-1}^2}{nn}$.

And $\left(1-\frac{\overline{n-1}^2}{nn}=\right)\frac{nn-\overline{n-1}^2}{nn}$ will be the probability of throwing one ace at leaſt in two throws.

QUESTION XXXI.

What is the probability of throwing an ace (or any one of the faces of the die) in three throws, that is, either at the firſt, ſecond, or third throw?

SOLUTION.

The probability of throwing the required face in two throws is $\frac{nn-\overline{n-1}^2}{nn}$, and the probability of miſſing it the two firſt throws is $\frac{\overline{n-1}^2}{nn}$ by queſtion 30.

Therefore the probability of miſſing it the two firſt throws, and throwing it the third, will be $\left(\frac{\overline{n-1}^2}{nn}\times\frac{1}{n}=\right)$ $\frac{\overline{n-1}^2}{n^3}$.

Therefore, the probability of throwing the required face, either at the firſt, ſecond, or third throw, will be $\frac{nn-\overline{n-1}^2}{nn}+\frac{\overline{n-1}^2}{n^3}$,

Or $\frac{n^3-n\times\overline{n-1}^2}{n^3}+\frac{\overline{n-1}^2}{n^3}$:

But $n\times\overline{n-1}^2-\overline{n-1}^2=\overline{n-1}^3$,

$$\text{For}\begin{cases}\text{From} & n^3-2nn+n,\\ \text{Take} & nn-2n+1;\\ \text{Remains} & n^3-3nn+3n-1.\end{cases}$$

Therefore $\frac{n^3-\overline{n-1}^3}{n^3}$ will be the probability required.

In this caſe $\left(\frac{216-125}{216}=\right)\frac{91}{216}$.

Otherwiſe, take $\left(\frac{n-1}{n}\times\frac{n-1}{n}\times\frac{n-1}{n}=\right)\frac{\overline{n-1}^3}{n^3}$, the probability of miſſing an ace three times ſucceſſively, from unity, and the remainder $\left(1-\frac{\overline{n-1}^3}{n^3}=\right)\frac{n^3-\overline{n-1}^3}{n^3}$ will be the probability of throwing an ace once or more, in three throws, as before.

COROL.

Hence, the probability of throwing one ace, or more, in m throws, may be found by ſubtracting $\left(\frac{\overline{n-1}^m}{n^3}\right)$ the probability of miſſing an ace m times, from unity; which being done the remainder $\left(1-\frac{\overline{n-1}^m}{n^m}=\right)\frac{n^m-\overline{n-1}^m}{n^m}$ is the ſaid probability.

QUESTION XXXII.

The probability of throwing one ace, and no more, in two throws is required?

SOLU-

SOLUTION.

From $\left(\frac{nn-\overline{n-1}^2}{nn}\right)$ the probability of throwing one ace, or more, take $\frac{1}{nn}$ the probability of throwing two aces, in two throws; and the remainder $\left(\frac{nn-\overline{n-1}^2}{nn}-\frac{1}{nn}=\frac{2n-1}{nn}-\frac{1}{nn}=\frac{2n-2}{nn}=\right)\frac{\overline{n-1}\times 2}{nn}$ will be the probability required. In this caſe $\left(\frac{5\times 2}{36}=\right)\frac{5}{18}$

QUESTION XXXIII.

What is the probability of throwing one ace, and no more, in three throws?

SOLUTION.

If an ace be thrown the firſt time, (the probability of which is $\frac{1}{n}$) then never an ace muſt be thrown in the next two Trials (the probability of which is $\frac{\overline{n-1}^2}{n^2}$).

And $\left(\frac{1}{n}\times\frac{\overline{n-1}^2}{n^2}=\right)\frac{\overline{n-1}^2}{n^3}$ is the probability of throwing in that order.

If an ace be miſſed the firſt throw (the probability of which is $\frac{n-1}{n}$) then but one ace muſt be thrown in the remaining two throws, (the probability of which is $\frac{\overline{n-1}\times 2}{nn}$) And $\left(\frac{n-1}{n}\times\frac{\overline{n-1}\times 2}{nn}=\right)\frac{\overline{n-1}^{2}\times 2}{nnn}$ is the probability of throwing in that order.

Therefore the whole probability of throwing but one ace, in three throws, will be $\left(\frac{\overline{n-1}^{2}}{n^{3}}+\frac{\overline{n-1}^{2}\times 2}{n^{3}}=\right)$ $\frac{\overline{n-1}^{2}\times 3}{nnn}$. In this caſe $\left(\frac{25\times 3}{216}=\right)\frac{25}{72}$.

QUESTION XXXIV.

The probability of throwing one ace, and no more, in four throws, is required?

SOLUTION.

The probability of throwing an ace the firſt throw, and miſſing it for the other three throws will be $\left(\frac{1}{n}\times\frac{\overline{n-1}^{3}}{n^{3}}=\right)\frac{\overline{n-1}^{3}}{n^{4}}$.

And the probability of miſſing an ace the firſt time, and throwing but one in the three following throws will be $\left(\frac{n-1}{n}\times\frac{\overline{n-1}^{2}\times 3}{n^{3}}=\right)\frac{\overline{n-1}^{3}\times 3}{n^{4}}$.

Therefore the whole probability of throwing one ace, and no more, in 4 throws will be $\left(\frac{\overline{n-1}^{3}}{n^{4}}+\frac{\overline{n-1}^{3}\times 3}{n^{4}}=\right)$ $\frac{\overline{n-1}^{3}\times 4}{n^{4}}$. In this case $\left(\frac{125\times 4}{1296}=\right)\frac{125}{324}$.

COROL.

Therefore the probability of throwing one ace, and no more, in m throws will be $\frac{\overline{n-1}^{m-1}\times m}{n^{m}}$.

QUESTION XXXV.

What is the probability of throwing two aces, or more, in three throws?

SOLUTION.

If $\left(\frac{\overline{n-1}^{3}}{n^{3}}\right)$ the probability of throwing never an ace in three throws, and $\left(\frac{\overline{n-1}^{2}\times 3}{n^{3}}\right)$ the probability of throwing but one ace in three throws, be both taken from unity, there will remain $\left(1-\frac{\overline{n-1}^{3}}{n^{3}}-\frac{\overline{n-1}^{2}\times 3}{n^{3}}=\right)$ $\frac{n^{3}-\overline{n-1}^{3}-\overline{n-1}^{2}\times 3}{n^{3}}$, the probability of throwing two aces, or more, in three throws.

In this caſe $\left(\frac{216-125-25\times3}{216}=\frac{216-200}{216}=\frac{16}{216}=\right)\frac{2}{27}$.

QUESTION XXXVI.

What is the probability of throwing two aces, or more, in four throws?

SOLUTION.

From unity, take $\left(\frac{\overline{n-1}^{4}}{n^{4}}+\frac{\overline{n-1}^{3}\times4}{n^{4}}\right)$ the ſum of the reſpective probabilities of having, (in four throws) never an ace, and but 1 ace; And the remainder $\left(1-\frac{\overline{n-1}^{4}}{n^{4}}-\frac{\overline{n-1}^{3}\times4}{n^{4}}=\right)\frac{n^{4}-\overline{n-1}^{4}-\overline{n-1}^{3}\times4}{n^{4}}$ will be the probability of throwing two aces, or more, in 4 throws.

In this caſe $\left(\frac{1296-625-125\times4}{1296}=\frac{1296-1125}{1296}=\frac{171}{1296}=\right)\frac{19}{144}$.

COROL.

Therefore the probability of throwing 2 aces, or more, in m throws will be $\frac{n^{m}-\overline{n-1}^{m}-\overline{n-1}^{m-1}\times m}{n^{m}}$.

QUESTION XXXVII.

The probability of throwing two aces, and no more, in three throws is required?

SOLUTION.

From $\left(\frac{n^3-\overline{n-1}^3-\overline{n-1}^2\times3}{n^3}\right)$ the probability of throwing two aces, or more, in three throws, take $\frac{1}{n^3}$ the probability of throwing three aces in three throws, and the remainder $\left(\frac{n^3-\overline{n-1}^3-\overline{n-1}^2\times3}{n^3}-\frac{1}{n^3}\right)$ is the probability of throwing two aces, and no more, in three throws.

But $\left(\frac{n^3-\overline{n-1}^3-\overline{n-1}^2\times3-1}{n^3}=\frac{3n-3}{n^3}=\right)\frac{\overline{n-1}\times3}{n^3}$,

$$
\begin{array}{lrcl}
\text{For} & n^3 & = & n^3 \\
\text{And} & \left\{\begin{array}{r} \overline{n-1}^3 \\ \overline{n-1}^2\times3 \\ 1 \end{array}\right. & \begin{array}{c} = \\ = \\ = \end{array} & \begin{array}{r} \overline{n^3-3n^2+3n-1}, \\ 3n^2-6n+3, \\ +1, \end{array} \\
\hline
\text{Sum} & & & n^3 \quad -3n+3; \\
\hline
\text{Remainder} & & & 3n-3: \\
\hline
\end{array}
$$

Therefore $\frac{\overline{n-1}\times3}{n^3}$ will be the probability of throwing two aces, and no more, in three throws.

QUESTION XXXVIII.

The probability of throwing two aces, and no more, in 4 throws, is required?

SOLUTION.

If an ace be thrown the firſt throw (the probability of which is $\frac{1}{n}$) then but one ace muſt be thrown in the three following throws (the probability of which is $\frac{\overline{n-1}^2 \times 3}{n^3}$ by queſt. 33): And the probability of throwing it in this order, will be $\left(\frac{1}{n} \times \frac{\overline{n-1}^2 \times 3}{n^3} = \right) \frac{\overline{n-1}^2 \times 3}{n^4}$.

But if an ace be miſſed the firſt throw (the probability of which is $\frac{n-1}{n}$) then two aces, and no more, muſt be thrown in the remaining three throws (the probability of which is $\frac{\overline{n-1} \times 3}{n^3}$ by queſt· 37); and the probability of throwing it in this order will be $\left(\frac{n-1}{n} \times \frac{\overline{n-1} \times 3}{n^3} = \right)$ $\frac{\overline{n-1}^2 \times 3}{n^4}$.

Therefore the whole probability of throwing two aces, and no more, in four throws will be $\left(\frac{\overline{n-1}^2 \times 3}{n^4} + \frac{\overline{n-1}^2 \times 3}{n^4} = \right) \frac{\overline{n-1}^2 \times 6}{n^4}$; In this caſe $\left(\frac{25 \times 6}{1296} = \right) \frac{25}{216}$.

QUESTION XXXIX.

The probability of throwing two aces, and no more, in five throws, is required?

SOLUTION.

The probability of throwing an ace the firſt throw, and throwing but one ace in four throws after, will be $\left(\frac{1}{n}\times\frac{\overline{n-1}^3\times4}{n^4}=\right)\frac{\overline{n-1}^3\times4}{n^5}$, by queſt. 34.

And the probability of miſſing an ace the firſt throw, and throwing it twice, only, in the 4 ſucceeding throws is $\left(\frac{n-1}{n}\times\frac{\overline{n-1}^2\times6}{n^4}=\right)\frac{\overline{n-1}^3\times6}{n^5}$ by queſt. 38.

Therefore the whole probability of throwing two aces, and no more, in five throws will be $\left(\frac{\overline{n-1}^3\times4}{n^5}+\frac{\overline{n-1}^3\times6}{n^5}=\right)\frac{\overline{n-1}^3\times10}{n^5}$; In this caſe $\left(\frac{125\times10}{7776}=\right)$ $\frac{625}{3888}$.

COROL.

Since the probability of throwing two aces, and no more, in

$$\left\{\begin{matrix} 3 \\ 4 \\ 5 \end{matrix}\right\}\text{Throws is}\left\{\begin{matrix} \dfrac{\overline{n-1}\times 3}{n^3} & \text{by queſt. } 37. \\ \dfrac{\overline{n-1}^{\,2}\times 6}{n^4} & 38. \\ \dfrac{\overline{n-1}^{\,3}\times 10}{n^5} & 39. \end{matrix}\right.$$

Therefore the probability of throwing two aces, and no more, in *m* throws will be $\overline{n-1}^{\,\overline{m-2}}\times\frac{\overline{m\cdot m-1}}{1\cdot 2}$.

SCHOLIUM. The chance of throwing any two faces at one throw, with two dice, is the ſame, with that of throwing the ſame faces, at two operations, with a ſingle die; as will appear from what follows.

All the different throws, upon two dice, are repreſented in the following table; where the numbers 1. 2. 3. 4. 5. 6. denote the ſeveral faces of the one die, and *1. 2. 3. 4. 5. 6.* the faces of the other die.

1.1	2.1	3.1	4.1	5.1	6.1
1.2	2.2	3.2	4.2	5.2	6.2
1.3	2.3	3.3	4.3	5.3	6.3
1.4	2.4	3.4	4.4	5.4	6.4
1.5	2.5	3.5	4.5	5.5	6.5
1.6	2.6	3.6	4.6	5.6	6.6

From which table it appears,

1ſt. That $nn=36$ is the whole number of chances on two dice; and conſequently that 36 will be the common denominator of all fractions, that expreſs probabilities concerning them.

2d. That the probability of throwing one ace, or more, with two dice, will be $\frac{11}{36}$; the ſame as that of throwing one ace, or more, at two throws, with one die, ſee queſt. 30; for, in the table, one or more aces will be found in every one of the ſix uppermoſt ſquares, and in five other ſquares in the left hand column.

3d. Alſo the probability of throwing one ace, and no more, in two throws with one die, which by queſt. 32 is $\frac{10}{36}$, will appear to be equal to the probability of throwing only one ace with two dice, which is alſo $\frac{10}{36}$;

for of the 11 chances, for one ace, or more, above-quoted from the table, there is one for two aces.

4th. That the probability of throwing an ace, or a deux, will be $\left(\frac{20}{36}=\right)\frac{5}{9}$; for an ace will be found in every one of the 6 ſquares of the upper line, a deux will be found in every one of the 6 ſquares of the ſecond line, an ace will be found in the four remaining ſquares of the left hand column, and a deux in the 4 remaining ſquares of the next column thereto.

5th. In like manner the probability of throwing ace, deux, or trè, will be $\left(\frac{27}{36}=\right)\frac{3}{4}$; and that of throwing ace, deux, trè, or quator $\left(\frac{32}{36}=\right)\frac{8}{9}$.

QUESTION XL.

The reſpective probabilities of throwing any number of points, with two dice, are required?

SOLUTION.

This queſtion may be readily anſwered, by diſpoſing the numbers of the above table, in the following order.

N° of Points.	How they may be thrown.						Chan. thereof.
2	1.1						1
3	1.2	2.1					2
4	1.3	2.2	3.1				3
5	1.4	2.3	3.2	4.1			4
6	1.5	2.4	3.3	4.2	5.1		5
7	1.6	2.5	3.4	4.3	5.2	6.1	6
8	2.6	3.5	4.4	5.3	6.2		5
9	3.6	4.5	5.4	6.3			4
10	4.6	5.5	6.4				3
11	5.6	6.5					2
12	6.6						1
						Total of the Chances	36

That is the probability of throwing with two dice $\left\{\begin{matrix}5\\6\\7\end{matrix}\right\}$ points, &c. is $\left\{\begin{matrix}\frac{4}{36}\\\frac{5}{36}\\\frac{6}{36}\end{matrix}\right\}$ &c.

SCHOLIUM. It may be worth obſerving, that if the multinomial, $r+r^2+r^3+r^4+r^5+r^6$, be ſquared (that is multiplied by itſelf) the numeral coefficients, annexed to each power of r in the product, will be the number of chances, for the ſame number of points, as is expreſſed by the Index of the power, to which they are annexed. See the operation:

$$\begin{array}{ccccccccccc}
r & +r^2 & +r^3 & +r^4 & +r^5 & +r^6 & & & & & \\
r & +r^2 & +r^3 & +r^4 & +r^5 & +r^6 & & & & & \\
\hline
r^2 & +r^3 & +r^4 & +r^5 & +r^6 & +r^7 & & & & & \\
 & +r^3 & +r^4 & +r^5 & +r^6 & +r^7 & +r^8 & & & & \\
 & & +r^4 & +r^5 & +r^6 & +r^7 & +r^8 & +r^9 & & & \\
 & & & +r^5 & +r^6 & +r^7 & +r^8 & +r^9 & +r^{10} & & \\
 & & & & +r^6 & +r^7 & +r^8 & +r^9 & +r^{10} & +r^{11} & \\
 & & & & & +r^7 & +r^8 & +r^9 & +r^{10} & +r^{11} & +r^{12} \\
\hline
r^2 & +2r^3 & +3r^4 & +4r^5 & +5r^6 & +6r^7 & +5r^8 & +4r^9 & +3r^{10} & +2r^{11} & +r^{12}
\end{array}$$

QUESTION XLI.

A, being at play at back-gammon, is obliged to make a blot; now his throw is ſuch, that he can make it, either where his adverſary, *B*, can take it up with a ſingle ace, or elſe where he can take it up by throwing ſeven in any manner: The queſtion is, where he ſhould make the blot?

SOLUTION.

Becauſe the number of chances for throwing one ace or more is 11, and the number of chances for throwing 7, in any manner, are but 6; Therefore, it will be ſafeſt to make the blot, where it may be taken up by throwing 7.

QUESTION XLII.

Whether is it ſafer, to make a blot at back-gammon, where it can be taken up by an ace, or where it can be taken up by a trè?

SOLUTION.

The number of chances for throwing one ace or more, and thoſe for throwing one trè or more, are each eleven; but then, there are two chances for throwing deux ace, or three; therefore it will be ſafer to make the blot, where it can be taken up, only, by an ace.

The following table will ſhew the chances of taking up a ſingle blot, howſoever ſituated.

No. of points to hit.	Chances.	Total Chances.
1	11	11
2	11 + 1	12
3	11 + 2	13
4	11 + 3	14
5	11 + 4	15
6	11 + 5	16

No. of points to hit.	Chances.	Total Chances.
7	6	6
8	5	5
9	4	4
10	3	3
11	2	2
12	1	1

COROL.

Hence, if the blot is liable to be hit by any one face of the die, the mean probability of hitting it will be $\left(\frac{11+16}{2\times 30}=\frac{27}{72}=\right)\frac{3}{8}$ nearly.

QUESTION XLIII.

If two blots be made at backgammon, ſo as to be hit by two different faces of the die, what is the probability of hitting one, or both of them ?

SOLUTION.

By the firſt given table, it will appear that the probability of throwing one, or more, of any two given faces is $\frac{20}{36}$.

But beſides that; one or both the blots may be hit by the two dice at length; and the probability of that, is different, according to the number of points that will hit them, as in the following table:

Faces to hit.	Chances.	Total Chances.
1·2	20+1	21
1·3	20+2	22
1·4	20+3	23
1·5	20+4	24
1·6	20+5	25
2·3	20+1+2	23
2·4	20+1+3	24
2·5	20+1+4	25

Faces to hit.	Chances.	Total Chances.
2·6	20+1+5	26
3·4	20+2+3	25
3·5	20+2+4	26
3·6	20+2+5	27
4·5	20+3+4	27
4·6	20+3+5	28
5·6	20+4+5	29

COROL.

Hence the probability of hitting two ſuch blots will, at a medium, be $\left(\frac{21+29}{2\times36}=\right)\frac{25}{36}$.

QUESTION XLIV.

If there be three blots, ſo ſituated as to be hit by three different faces of the die, the probability of hitting one or more of them is required?

SOLUTION.

The firſt table will give the probability of hitting one, or more, of the blots with a ſingle face, or faces; but beſide that, there will be the probability of hitting one, or more, of the blots with two dice, at length; the leaſt of which will be, when the given faces are 1, 2, 3, which have $(2+1=)$ 3 ſuch chances; and the greateſt, when the given faces are 4, 5, 6, which have $(3+4+5=)$ 12 ſuch chances, the medium of which, *viz.* $\left(\frac{3+12}{2}=\right)\frac{15}{2}$ being added to 27, makes the whole probability, about $\left(27+\frac{15}{2}=\right)\frac{69}{2}$, which divided by the common denominator 36 becomes $\left(\frac{69}{72}=\right)\frac{23}{24}$.

COROL.

Hence, if a player at backgammon makes 3 blots, which are ſeverally within reach of being hit by a ſingle face of the die, it is almoſt a certainty that one of them, at leaſt, will be hit.

QUES-

QUESTION XLV.

Suppoſe two purſes, each containing n counters; whereof a are white, and b black: If a perſon draw a counter out of each purſe, what probability hath he to draw one white counter, and no more?

SOLUTION.

The probability of his drawing a black counter out of the firſt purſe, and a white counter out of the ſecond will (by corol. to queſt. 29) be $\left(\overline{1-\frac{a}{a+b}}\times\frac{a}{a+b}=\frac{a}{a+b}-\frac{aa}{\overline{a+b}^2}=\frac{\overline{a+b}\times a-aa}{\overline{a+b}^2}=\frac{aa+ab-aa}{\overline{a+b}^2}=\right)$ $\frac{ab}{\overline{a+b}^2}$.

And the probability of his drawing a white counter out of the firſt purſe, and a black counter out of the ſecond, will, by a like argument, be the ſame, *viz.* $\frac{ab}{\overline{a+b}^2}$.

Therefore the probability required will be $\left(\frac{ab}{\overline{a+b}^2}+\frac{ab}{\overline{a+b}^2}=\right)$ $\frac{2ab}{\overline{a+b}^2}$.

COROL.

Hence, if a and b ſeverally repreſent the number of chances, for the happening and failing of an event, at one trial; then ſhall one, or more, of the terms of the ſecond power of the binomial $\overline{a+b}^2$ be the numerators of the fractions, which will expreſs the probabilities of all the varieties that can poſſibly happen, concerning two ſuch events in two trials. And the power itſelf $\overline{a+b}^2$ will be the common denominator of the ſaid fractions.

For inſtance, the probability, in two trials, of the happening of

$$\left.\begin{array}{l}\text{Two events}\\[2ex] \text{Only one of them}\\[2ex] \text{Neither of them}\end{array}\right\}\text{will be}\left\{\begin{array}{c}\dfrac{aa}{\overline{a+b}^2}\\[1ex] \dfrac{2ab}{\overline{a+b}^2}\\[1ex] \dfrac{bb}{\overline{a+b}^2}\end{array}\right\}\begin{array}{r}\text{Queſt. 28.}\\[2ex] \text{45.}\\[2ex] \text{Corol. 29.}\end{array}$$

The ſum of which three probabilities $\frac{aa}{\overline{a+b}^2}$ $\frac{2ab}{\overline{a+b}^2}$ and $\frac{bb}{\overline{a+b}^2}$, *viz.* $\frac{aa+2ab+bb}{aa+2ab+bb}$ is unity, as it evidently ought to be.

Again the probability of one or both of the events happening will be $\left(\frac{aa+2ab}{\overline{a+b}^2}=\right)\frac{\overline{a+b}^2-bb}{\overline{a+b}^2}$; And the probability that both of them will not happen, will be $\left(\frac{2ab+bb}{\overline{a+b}^2}=\right)\frac{\overline{a+b}^2-aa}{\overline{a+b}^2}$.

QUES-

QUESTION XLVI.

If there be three purſes, each containing *a*, white, and *b*, black counters; and if a perſon draw one counter out of each purſe, what probability is there that they ſhall be all white?

SOLUTION.

If he draws a white counter out of the firſt purſe (the probability of which is $\frac{a}{a+b}$) then he muſt draw two white counters out of the two remaining purſes (the probability of which is $\frac{aa}{\overline{a+b}^2}$): but neither of thoſe events will be effectual without the happening of the other; and therefore $\left(\frac{a}{a+b} \times \frac{aa}{\overline{a+b}^2} =\right) \frac{a^3}{\overline{a+b}^3}$ will be the probability required.

COROL.

In like manner the probability of drawing three black counters, or of failing in each attempt will be $\frac{bbb}{\overline{a+b}^3}$.

QUES-

QUESTION XLVII.

If a perſon draws out of three purſes, as in the laſt queſtion, what is the probability that two of the counters drawn, and no more, ſhall be white?

SOLUTION.

If he draws a black counter the firſt time (the probability of which is $\frac{b}{a+b}$) then he muſt draw two white counters out of the two remaining purſes (the probability of which is $\frac{aa}{\overline{a+b}^{2}}$), and therefore the probability of ſucceeding, by drawing in that manner will be $\left(\frac{b}{a+b} \times \frac{aa}{\overline{a+b}^{2}} =\right) \frac{aab}{\overline{a+b}^{3}}$.

If he draws a white counter the firſt time (the probability of which is $\frac{a}{a+b}$) then he muſt draw, only, one white counter out of the two remaining purſes (the probability of which is $\frac{2ab}{\overline{a+b}^{2}}$ by queſt. 45); And therefore the probability of ſucceeding, in this manner, will be $\left(\frac{a}{a+b} \times \frac{2ab}{\overline{a+b}^{2}} =\right) \frac{2aab}{\overline{a+b}^{3}}$. Therefore the whole probability of the thing required will be $\left(\frac{aab}{\overline{a+b}^{3}} + \frac{2aab}{\overline{a+b}^{3}} =\right) \frac{3aab}{\overline{a+b}^{3}}$.

COROL.

Hence the probability of drawing two white counters, more, will be the ſum of the reſults of the two laſt eſtions, viz. $\frac{a^3+3aab}{\overline{a+b}^3}$.

QUESTION XLVIII.

If a perſon draws out of the three purſes, as in the wo laſt queſtions, what is the probability that one white ounter, and no more, ſhall be drawn?

SOLUTION.

If he draws a white counter the firſt time; then, at he other two trials, he muſt draw two black ones; the robability of doing both which is $\left(\frac{a}{a+b}\times\frac{bb}{\overline{a+b}^2}=\right)$ $\frac{abb}{\overline{a+b}^3}$.

If he draws a black counter at the firſt trial; then, t the other two trials, he muſt draw one black and one white counter; the probability of doing both which is $\left(\frac{b}{a+b}\times\frac{2ab}{\overline{a+b}^2}=\right)\frac{2abb}{\overline{a+b}^3}$.

Therefore the probability required will be $\left(\frac{abb}{\overline{a+b}^3}+\frac{2abb}{\overline{a+b}^3}=\right)\frac{3abb}{\overline{a+b}^3}$.

COROL.

COROL. I.

The probability of drawing one, or two white counters, neither leſs than one, nor more than two, will be $\left(\frac{3aab}{\overline{a+b}^3}+\frac{3abb}{\overline{a+b}^3}=\right)\frac{3ab}{\overline{a+b}^2}$.

COROL. II.

The probability of drawing one, two, or three white counters, that is, not leſs than one, will be $\left(\frac{a^3}{\overline{a+b}^3}+\frac{3aab}{\overline{a+b}^3}+\frac{3abb}{\overline{a+b}^3}=\right)\frac{\overline{a+b}^3-b^3}{\overline{a+b}^3}$.

COROL. III.

The probability of drawing none, or, at moſt, but one, white counter, will be $\frac{3abb+b^3}{\overline{a+b}^3}$.

COROL. IV.

And the probability of drawing none, one, or at moſt but two, white counters, will be $\frac{\overline{a+b}^3-a^3}{\overline{a+b}^3}$.

COROL. V.

Hence it appears, that one or more of the terms of the binomial $\overline{a+b}$, raiſed to the third power, will be he numerators of thoſe fractions, which expreſs the proabilities of all the varieties that can poſſibly happen in hree trials; concerning events, the number of chances, r the happening or failing of one of which, are reſpecively *a* or *b*; and that the common denominator of all hoſe fractions will be $\overline{a+b}^3$ the power itſelf.

For inſtance, the probability, in three trials, of the happening of,

The three events	will be	$\frac{a^3}{\overline{a+b}^3}$	by	Queſt. 46.
Only two of them		$\frac{3aab}{\overline{a+b}^3}$		47.
Only one of them		$\frac{3abb}{\overline{a+b}^3}$		48.
Neither of them		$\frac{b^3}{\overline{a+b}^3}$		Corol. to 46.

The ſum of which four probabilities, *viz.* $\frac{a^3+3aab+3abb+b^3}{\overline{a+b}^3}$ is unity, as it manifeſtly ſhould be.

GENERAL COROLLARIES.

1*ſt*. Therefore all the queſtions that can poſſibly be aſked, concerning the happening, or failing, of any number of events, in *m* trials will (if *a* expreſſes the chances for happening, and *b* the chances for failing) be anſwered, by the aſſiſtance of one, or more, of the terms of the binomial $\overline{a+b}^m$, as a numerator, and the whole binomial, as a denominator.

That is to ſay, the probability of the happening of

$$\left.\begin{array}{c} m\\ m-1\\ m-2\\ m-3\\ \&c.\\ \text{to}\\ 3\\ 2\\ 1\\ 0\end{array}\right\}\text{of the events will be}\left\{\begin{array}{l} a^m\\ ma^{m-1}b\\ \frac{m\cdot\overline{m-1}}{1\cdot2}a^{m-2}b^2\\ \frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{1\cdot2\cdot3}a^{m-3}b^3\\ \&c.\text{ to}\\ \frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{1\cdot2\cdot3}b^{m-3}a^3\\ \frac{m\cdot\overline{m-1}}{1\cdot2}b^{m-2}a^2\\ mb^{m-1}a\\ b^m\end{array}\right\}\text{each ſeverally divided by }\overline{a+b}^m$$

2. The probability of the happening of m, $m-1$, $m-2$, or at feweſt of but $m-n$, ſuch events will be

$$\frac{a^m+ma^{m-1}b+\frac{m\cdot\overline{m-1}}{1\cdot2}a^{m-2}b^2+\frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{1\cdot2\cdot3}a^{m-3}b^3}{\overline{a+b}^m}$$

the numerator of which fraction muſt conſiſt of $n+1$ terms.

And the probability of the happening of, at leaſt, one ſuch event, will be

$$\frac{\overline{a+b}^m-b^m}{\overline{a+b}^m}$$

3. The probability of the happening of, neither, or but one, or two, or at moſt of but n, ſuch events will be

$$\frac{b^m + mb^{m-1}a + \frac{m\cdot\overline{m-1}}{1\cdot2}b^{m-2}a^2 + \frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{1\cdot2\cdot3}b^{m-3}a^3}{\overline{a+b}^m}$$

the numerator of which fraction muſt conſiſt of $n+1$ terms.

And the probability of the happening, at moſt, but $m-1$ ſuch events will be

$$\frac{\overline{a+b}^m - a^m}{\overline{a+b}^m}$$

QUESTION XLIX.

In a lottery, whereof the number of blanks is to the number of prizes as 39 to 1 (ſuch was the lottery of the year 1720) how many tickets muſt be purchaſed, that the buyer may have an equal chance for one or more prizes?

SOLUTION.

The probability of having one prize, or more, in m tickets, in a lottery wherein the blanks are to the prizes as 39 to 1, is the ſame, with that of throwing one ace, or more, in m throws, with a die that has $(1+39=)$ 40 faces: Putting therefore $40=n$, the ſaid probability will (by corol. queſt. 31.) be $\frac{n^m - \overline{n-1}^m}{n^m}$,

Or (if $1=a$ and $39=b$) $\frac{\overline{a+b}^m - b^m}{\overline{a+b}^m}$. And the probability of miſſing $\frac{b^m}{\overline{a+b}^m}$. But, by the queſtion, there is to be an equal chance for the having, or miſſing, one, or more prizes: Now, if the certainty of having or miſſing one, or more, prizes be denoted by unity, then the probabilities of an equal chance for having, or miſſing, one, or more, of them, will each of them be denoted by $\frac{1}{2}$.

Therefore $$\frac{\overline{a+b}^m - b^m}{\overline{a+b}^m} = \frac{b^m}{\overline{a+b}^m} = \frac{1}{2}$$

Whence $$2b^m = \overline{a+b}^m.$$

Which in logarithms will be $\Big\}$ $L.\,2 + m\,L.b = m \times L.\overline{a+b}$

Or $$Log.\,2. = m \times L.\overline{a+b} - m \times L.b$$

Th. $$\frac{Log.\,2}{L.\overline{a+b} - L.b} = m$$

In this caſe, $L.\,\overline{a+b} = L.40 = 1{,}60206$
$L.\quad b = L.39 = 1{,}59106$

$$\begin{array}{r} Log.\ \text{of}\ 2 \\ L.\overline{a+b} - L.b. = 0{,}01100)0{,}30103)27{,}36 = m \end{array}$$

Therefore the number of tickets muſt be greater than 27.

QUESTION L.

In a pack of 26 cards, 13 of which are black, and 13 red: If m cards be dealt, how many is there an equal chance of being red?

SOLU-

SOLUTION.

‘ If the number of chances for the happening of the event be denoted by a, and thoſe for its failing by b,

Then b^m,

$$b^m + m b^{m-1} a,$$

$$b^m + m b^{m-1} a + \frac{m \cdot m-1}{1 \cdot 2} b^{m-2} a^2,$$

&c.

being ſeverally divided by $\overline{b+a}^m$, will expreſs the probability of its not happening

&c. thrice, twice, once,

in m trials.

‘ And becauſe the queſtion requires, how many times the event will happen, in m trials, upon an equality of chance, it will follow, that when the event is to happen

Once, b^m

Twice, $b^m + m b^{m-1} a$

Thrice, $b^m + m b^{m-1} a + \frac{m \cdot m-1}{1 \cdot 2} b^{m-2} a^2$

&c.

being ſeverally divided by $\overline{b+a}^m$ will be equal to $\frac{1}{2}$: See queſt. 49.

Or $b^m = \frac{1}{2} \times \overline{b+a}^m$;

$$b^m + m b^{m-1} a = \frac{1}{2} \times \overline{b+a}^m;$$

$$b^m + m b^{m-1} a + \frac{m \cdot m-1}{1 \cdot 2} b^{m-2} a^2 = \frac{1}{2} \times \overline{b+a}^m;$$

&c. &c.

Whence the number of terms, which are equal to $\frac{1}{2} \times \overline{b+a}^m$ will be the anſwer to the queſtion.

This being premiſed, Let $a:b::1:p$;

Then $pa=b$; and $\frac{a}{b}\left(=\frac{a}{pa}=\right)\frac{1}{p}$.

Therefore if $1+\frac{1}{p}$ be ſubſtituted for $b+a$ the above expreſſions will become

$$\begin{aligned}
&1 &&= \overline{1+\tfrac{1}{p}}^m \times \tfrac{1}{2}.\\
&1+m\times\frac{1}{p} &&= \overline{1+\tfrac{1}{p}}^m \times \tfrac{1}{2}.\\
&1+m\times\frac{1}{p}+\frac{m\cdot m-1}{1\cdot2}\times\frac{1}{pp} &&= \overline{1+\tfrac{1}{p}}^m \times \tfrac{1}{2}.\\
&1+m\times\frac{1}{p}+\frac{m\cdot m-1}{1\cdot2}\times\frac{1}{pp}+\frac{m\cdot m-1\cdot m-2}{1\cdot2\cdot3}\times\frac{1}{ppp} &&= \overline{1+\tfrac{1}{p}}^m \times \tfrac{1}{2}.
\end{aligned}$$

&c.

Now, in the queſtion before us, where $a=b$, $p=1$; the expreſſion $\overline{1+\frac{1}{p}}\Big|^m$ will become $\overline{1+1}^m$. In which power, the ſeveral terms, at either extremity, are equal; and therefore $\overline{1+1}^m \times \frac{1}{2}$ will conſiſt of half the terms in $\overline{1+1}^m$.

But the number of terms in $\overline{1+1}^m = m+1$: Therefore the anſwer will be $\frac{m+1}{2}$.

COROL.

If r repreſent the number of times that the propoſed event is required to happen; then, when there is an equality of chance for its happening or miſſing, $r = \frac{m+1}{2}$ by the above queſtion. Therefore; m, the number of trials, in which it will be an equal chance whether the event ſhall happen r times or not, will be $= 2r-1$.

And therefore in a lottery, in which the number of prizes is equal to the number of blanks; if it be required to know how many tickets ſhould be bought, in order to have r, or more, prizes; the anſwer will be $2r-1$ tickets; that is, in order to have an equal chance to have 1, 2, 3, 4, 5, &c. (or more) prizes, there ſhould be bought 1, 3, 5, 7, 9, &c. tickets.

QUESTION LI.

In a pack of 39 cards, conſiſting of thirteen hearts, thirteen ſpades, and thirteen clubs: If m cards be dealt to me, how many may I, on an equality of chance, expect to be hearts?

SOLUTION.

If a, b and p repreſent the ſame as in the laſt queſtion, and r be the number required; then (becauſe there are two chances for a black card to be dealt, and but one for a red card) $b=2$, and $a=1$; and conſequently $p=2$.

Therefore $\left(1+\frac{1}{p}\right|^m \times \frac{1}{2}$ or $\left)\overline{\frac{p+1}{p}}\right|^m \times \frac{1}{2}=\overline{\frac{3}{2}}\Big|^m \times \frac{1}{2}$;

Alſo $1+m\times\frac{1}{2}+\frac{m\cdot\overline{m-1}}{1\cdot 2}\times\frac{1}{4}$ &c. to r terms $=$ $\overline{\frac{3}{2}}\Big|^m \times \frac{1}{2}$.

But we cannot here (as in the laſt queſtion) find the number of terms neceſſary to equal $\overline{\frac{3}{2}}\Big|^m \times \frac{1}{2}$; otherwiſe than by trials: In order to which

From	0,47712 = *Log.* 3,	
Take	0,30103 = *Log.* 2;	
Remains	0,17609 = *Log.* $\frac{3}{2}$:	

Now. when $m =$	Then $m \times L.\ \frac{3}{2}$ will be (0.17609 ×	And the N°. correſponding thereto, *viz.*	will be $\overline{\frac{3}{2}}\Big|^m$. Th.	will $=\overline{\frac{3}{2}}\Big|^m \times \frac{1}{2}$.
1;	1 =) ,17609;	1, 5,	0,75	
2;	2 =) ,35218;	2,25,	1,125	
4;	4 =) ,70436;	5,06,	2,53	
5;	5 =) ,88045;	7,59,	3,79	
7;	7 =) 1,23263;	17,08,	8,54	
8;	8 =) 1,40872;	25,62,	12,31	
10;	10 =) 1,76088;	57,66,	28,83	
11;	11 =) 1,93697;	86,48,	43,24	

And,

And, when $m =$	then	Terms of the power will be	
1,	1	1 - - - - - - - -	$= 1,0:$
2,	1	1 - - - - - - - -	$= 1,0:$
4,	2	$1 + 4\times\frac{1}{2}$ - - - - - -	$= 3,0:$
5,	2	$1 + 5\times\frac{1}{2}$ - - - - - -	$= 3,5:$
7,	3	$1 + 7\times\frac{1}{2} + \frac{7\cdot6}{1\cdot2}\times\frac{1}{4}$	$= 9,75:$
8,	3	$1 + 8\times\frac{1}{2} + \frac{8\cdot7}{1\cdot2}\times\frac{1}{4}$ - - -	$= 12,0:$
10,	4	$1 + 10\times\frac{1}{2} + \frac{10\cdot9}{1\cdot2}\times\frac{1}{4} + \frac{10\cdot9\cdot8}{1\cdot2\cdot3}\times\frac{1}{8}$	$= 32,25:$
11,	4	$1 + 11\times\frac{1}{2} + \frac{11\cdot10}{1\cdot2}\times\frac{1}{4} + \frac{11\cdot10\cdot9}{1\cdot2\cdot3}\times\frac{1}{8}$	$= 40,88:$

$$\text{Whence, if } m = \left\{\begin{array}{r}1,\\2,\\4,\\5,\\7,\\8,\\10,\\11,\end{array}\right\} \frac{m}{|3|2}\times\frac{1}{2} = \left\{\begin{array}{r}0,\ 75;\\1,125;\\2,\ 53;\\3,\ 79;\\8,\ 54;\\12,\ 81;\\28,\ 83;\\43,\ 24;\end{array}\right\} \text{And} \left\{\begin{array}{c}1\\1\\2\\2\\3\\3\\4\\4\end{array}\right\} \text{Terms, of } 1+m\times\frac{1}{2},\ \text{\&c. are,} = \left\{\begin{array}{r}1,\ 0.\\1,\ 0.\\3,\ 0.\\3,\ 5.\\9{,}75.\\12,\ 0.\\32,25.\\40,88.\end{array}\right.$$

$$\text{That is, when } m = \left\{\begin{array}{lll} \left\{\begin{array}{r}1,\\2,\end{array}\right\} & \text{then one term is} & \left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{ than}\\ \left\{\begin{array}{r}4,\\5,\end{array}\right\} & \text{then 2 terms are} & \left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{ than}\\ \left\{\begin{array}{r}7,\\8,\end{array}\right\} & \text{then 3 terms are} & \left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{ than}\\ \left\{\begin{array}{r}10,\\11,\end{array}\right\} & \text{then 4 terms are} & \left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{ than}\\ \text{\&c.} & & \text{\&c.}\end{array}\right\} \frac{m}{|3,2}\times\frac{1}{2}$$

And (putting $x =$ any number whatſoever)

When m is

$$\left\{\begin{matrix}3x+1\\3x+2\end{matrix}\right\}\text{then } x+1 \text{ terms are}\left\{\begin{matrix}\text{greater}\\\text{leſſer}\end{matrix}\right\}\text{than } \overline{\tfrac{3}{2}}|^{m}\times\tfrac{1}{2}.$$

But, when the chances for the events not happening r times, in m trials (which are ſignified by the above ſeries) are leſſer than $\frac{1}{2}$ the power; then the chances for the events happening are greater than it; Therefore, when $m=3x+2$; there will be more than an equal chance for the events happening $\overline{x+1}$ times; therefore, (putting $r=x+1$) we may argue as follows:

Since $m=3x+2$; Th. $\frac{m-2}{3}=x$:

And ſince $r=x+1$; $r-1=x$:

Therefore $\frac{m-2}{3}=r-1$,

Or $m-2=3r-3$;

Th. $\frac{m+1}{3}=r$.

That is, if m cards are dealt, it is more than an equal chance that there ſhould be $\frac{m+1}{3}$ hearts.

COROL.

Since $m-2=3r-3$;

Th. $m=3r-1$.

That is, in a lottery, where there are two blanks to a prize, if it be required to know, how many tickets ſhould be bought, in order to have an equal chance for r prizes, the anſwer will be $3r-1$ tickets; that is to ſay, in order to have an equal chance for obtaining 1, 2, 3, 4, 5, &

&c. prizes, there muſt be bought 2, 5, 8, 11, 14, &c. tickets.

QUESTION LII.

In a pack of 52 cards; conſiſting of 13 of each ſuit; if m cards be dealt to me, how many may I, on an equality of chance, expect to be trumps?

SOLUTION.

The ſymbols being retained, as before; Then becauſe there be three ſuits of blanks, to one ſuit of (trumps or) prizes; $b=3$, and $a=1$; Therefore $p=3$, and $\left.\overline{\frac{p+1}{p}}\right|^{m}\times\frac{1}{2}$ $=\left.\overline{\frac{4}{3}}\right|^{m}=\frac{1}{2}$:

Alſo $1+m\times\frac{1}{3}+\frac{m\cdot\overline{m-1}}{1\cdot 2}\times\frac{1}{9}$, &c. to r terms $=$ $\left.\overline{\frac{4}{3}}\right|^{m}\times\frac{1}{2}$.

Now from 0,60206 = *Log*. of 4.
Take 0,47712 = *Log*. of 3,

Remains 0,12494 = *Log*. of $\frac{4}{3}$:

And, when $m =$	Then, $m \times L.\frac{4}{3}$ will be (0,12494 ×	And the N°. correſponding thereto, *viz.* will be $\overline{\frac{4}{3}}\|^m$; Th.	$= \overline{\frac{4}{3}}\|^m \times \frac{1}{2}$.
2	2 =) 0, 24988;	1,78,	0,89
3.	3 =) 0, 37482;	2,37,	1,19
6.	6 =) 0, 74964;	5,62,	2,81
7.	7 =) 0, 87458;	7,49,	3,75
10,	10 =) 1, 24940;	17,76,	8,88
11,	11 =) 1, 37434;	23,68,	11,84
14,	14 =) 1, 74916;	56,13,	28,07
15,	15 =) 1, 87410;	74,83,	37,42

Alſo, when $m =$	Then,	Terms of the power will be
2	1	$1 = 1,0$:
3,	1	$1 = 1,0$:
6.	2	$1 + 6 \times \frac{1}{3} = 3,0$:
7,	2	$1 + 7 \times \frac{1}{3} = 3,33$:
10,	3	$1 + 10 \times \frac{1}{3} + \frac{10 \cdot 9}{1 \cdot 2} \times \frac{1}{9} = 9,33$:
11,	3	$1 + 11 \times \frac{1}{3} + \frac{11 \cdot 10}{1 \cdot 2} \times \frac{1}{9} = 10,78$:
14,	4	$1 + 14 \times \frac{1}{3} + \frac{14 \cdot 13}{1 \cdot 2} \times \frac{1}{9} + \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} \times \frac{1}{27} = 29,66$:
15	4	$1 + 15 \times \frac{1}{3} + \frac{15 \cdot 14}{1 \cdot 2} \times \frac{1}{9} + \frac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3} \times \frac{1}{27} = 34,39$:

That

$$\text{That is, when } m = \left\{\begin{array}{l} \left\{\begin{array}{l}2,\\3,\end{array}\right\}\text{then one term of the power is}\left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{than}\\ \left\{\begin{array}{l}6,\\7,\end{array}\right\}\text{then two terms are}\left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{than}\\ \left\{\begin{array}{l}10,\\11,\end{array}\right\}\text{then 3 terms are}\left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{than}\\ \left\{\begin{array}{l}14,\\15,\end{array}\right\}\text{then 4 terms are}\left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\}\text{than}\end{array}\right\} \overline{\tfrac{4}{3}}\Big|^{m}\times\tfrac{1}{2}$$

Th. when $m =$

$$\left\{\begin{array}{l}4x+2\\4x+3\end{array}\right\} \text{then } x+1 \text{ terms are} \left\{\begin{array}{c}\text{greater}\\\text{leſſer}\end{array}\right\} \text{than } \overline{\tfrac{4}{3}}\Big|^{m}\times\tfrac{1}{2}.$$

Therefore when $m=4x+3$; there will be ſomething more than an equal chance of the effects happening $x+1$ times.

Then ſince $m=4x+3$; $\frac{m-3}{4}=x$:

And ſince $r = x+1$; $r-1=x$:

Th. $r-1=\frac{m-3}{4}$; And $r=\left(\frac{m-3}{4}+1=\right)\frac{m+1}{4}$

That is; if m cards are dealt, it will be more than an equal chance, that there will be $\frac{m+1}{4}$ trumps. Therefore in the game of whiſt, where 13 cards are dealt, there is more than an equal chance for any particular perſons having $\frac{13+1}{4}=\frac{14}{4}$ trumps; And ſince this is more than an equal chance, if any player has but three trumps, or leſs; he may juſtly conclude that his partner has four trumps, or more.

COROL.

Since $r - 1 = \frac{m-3}{4}$,

Or $4r - 4 = m - 3$.

Th. $4r - 1 = m$.

That is; in a lottery, where there are three blanks to a prize, if it be required to know, how many tickets ſhould be bought, in order to have an equal chance to have *r* prizes, the anſwer will be $4r-1$ tickets; thus, in order to have 1, 2, 3, 4, 5, *&c.* prizes, there muſt be bought 3, 7, 11, 15, 19, *&c.* tickets.

QUESTION LIII.

In a lottery, which has four blanks to one prize; if I purchaſe *m* tickets, how many prizes may I expect, on an equality of chance?

SOLUTION.

The ſymbols being retained as before; then, in this caſe $b = 4$ and $a = 1$; Therefore $p = 4$, and $\overline{\frac{p+1}{p}}\Big|^{m} \times \frac{1}{2} = \overline{\frac{5}{4}}\Big|^{m} \times \frac{1}{2}$.

Alſo $1 + m \times \frac{1}{4} + \frac{m \cdot \overline{m-1}}{1 \cdot 2} \times \frac{1}{16}$ *&c.* $(r) = \overline{\frac{5}{4}}\Big|^{m} \times \frac{1}{2}$.

2

Now from 0,69897 = *Log.* of 5.
Take 0,60206 = *Log.* of 4.

Remains 0,09691 the *Log.* of $\frac{5}{4}$:

And, when m =	$m \times L. \frac{5}{4}$ will be (0,09691 ×	And the N°. correſponding thereto, *viz.*	will be $\overline{\frac{5}{4}}^{m}$; Th.	$= \overline{\frac{5}{4}}^{m} \times \frac{1}{2}$.
3,	3=)0,29073;	1,95,	0,98	
4,	4=)0,38764;	2,44;	1,22	
8,	8=)0,77528;	5,96,	2,98	
9,	9=)0,87219;	7,45.	3,73	
13	13=)1,25983;	18,19.	9,10	
14.	14=)1,35674;	22,74,	11,37	
18,	18=)1,74438;	55,51,	27,76	
19,	19=)1,84129;	69,39.	34,70	

Alſo, when m =

	Then	Terms of the power, *viz.*
3,	1	$1 = 1,\ 0$:
4,	1	$1 = 1,\ 0$:
8,	2	$1 + 8 \times \frac{1}{4} = 3,\ 0$:
9,	2	$1 + 9 \times \frac{1}{4} = 3{,}25$:
13,	3	$1 + 13 \times \frac{1}{4} + \frac{13 \cdot 12}{1 \cdot 2} \times \frac{1}{16} = 9{,}125$:
14,	3	$1 + 14 \times \frac{1}{4} + \frac{14 \cdot 13}{1 \cdot 2} \times \frac{1}{16} = 10{,}19$:
18,	4	$1 + 18 \times \frac{1}{4} + \frac{18 \cdot 17}{1 \cdot 2} \times \frac{1}{16} + \frac{18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3} \times \frac{1}{64} = 27{,}81$:
19,	4	$1 + 19 \times \frac{1}{4} + \frac{19 \cdot 18}{1 \cdot 2} \times \frac{1}{16} + \frac{19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3} \times \frac{1}{64} = 31{,}58$:

That

$$\text{That is, when } m = \left\{\begin{array}{l} \left\{\begin{matrix}3,\\ 4,\end{matrix}\right\} \begin{matrix}\text{Then one term of}\\ \text{the power is}\end{matrix} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than} \\ \left\{\begin{matrix}8,\\ 9,\end{matrix}\right\} \text{then 2 terms are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than} \\ \left\{\begin{matrix}13,\\ 14,\end{matrix}\right\} \text{then 3 terms are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than} \\ \left\{\begin{matrix}18,\\ 19,\end{matrix}\right\} \text{then 4 terms are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than} \end{array}\right\} \overline{\tfrac{5}{4}}\Big|^{m} \times \tfrac{1}{2}$$

Th. when $m =$

$$\left\{\begin{matrix}5x+3\\ 5x+4\end{matrix}\right\} \begin{matrix}\text{then } x+1 \text{ terms of}\\ \text{it are}\end{matrix} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than } \overline{\tfrac{5}{4}}\Big|^{m} \times \tfrac{1}{2}.$$

Therefore when $m = 5x+4$, there will be ſomething more than an equal chance of the required effects happening $x+1$ times.

Then ſince $m = 5x+4$; $\frac{m-4}{5} = x$:

And ſince $r = x+1$; $r-1 = x$:

Th. $r-1 = \frac{m-4}{5}$; And $r = \left(\frac{m-4}{5}+1=\right)\frac{m+1}{5}$.

That is, $\frac{m+1}{5}$ prizes may, on an equality of chance, be expected in m tickets.

COROL.

Since $r-1 = \frac{m-4}{5}$;

Th. $5r-5 = m-4$,

And $5r-1 = m$.

That is, in ſuch a lottery, if it be required to know, how many tickets ſhould be bought, in order to have an equal chance for r prizes, the anſwer will be $5r-1$ tickets; therefore, in order to have 1, 2, 3, 4, 5, &c. prizes, there ſhould be purchaſed 4, 9, 14, 19, 24, &c. tickets.

QUESTION LIV.

If a perſon, playing with a ſingle die, determines to caſt it m times; how many times, out of that number, may he, on an equality of chance, undertake to caſt an ace: Or, (which is the ſame thing) if he caſts m dice at once, how many of them may he, on an equality of chance, expect to be aces.

SOLUTION.

The ſymbols being retained as before; then, in this caſe $b=5$; and $a=1$; Th: $p=5$; and $\overline{\frac{p+1}{p}}\Big|^m \times \frac{1}{2} = \overline{\frac{6}{5}}\Big|^m \times \frac{1}{2}$.

Alſo $1+m\times\frac{1}{5}+\frac{m\cdot\overline{m-1}}{1\cdot 2}\times\frac{1}{25}$ &c. $(r)=\overline{\frac{6}{5}}\Big|^m\times\frac{1}{2}$.

Now from *Log.* $6 = 0{,}77815$,
Take *Log.* $5 = 0{,}69897$,

Remains *Log.* $\frac{6}{5} = 0{,}07918$;

And,

And, when $m=$	$m \times L.\ \frac{6}{5}$ will be $(0{,}07918 \times$	And the number corresponding, *viz.*	will be $\overline{\frac{6}{5}}\vert^{m} \times \frac{1}{2}$; Th.
3,	$3=)\ 0{,}23754$;	1,73;	0,87.
4,	$4=)\ 0{,}31672$;	2,07,	1,04.
9,	$9=)\ 0{,}71262$;	5,16,	2,58,
10,	$10=)\ 0{,}79180$;	6,19,	3,10,
15,	$15=)\ 1{,}18770$;	15,41,	7,71,
16,	$16=)\ 1{,}26688$;	18,49,	9,25,
21,	$21=)\ 1{,}66278$;	46,00,	23,00,
22,	$22=)\ 1{,}74196$;	55,20,	27,60,
			$=\overline{\frac{6}{5}}\vert^{m} \times \frac{1}{2}$.

Also, when $m=$	Then	Terms of the power, *viz.*	
3,	1	1	$= 1,\ 0$:
4,	1	1	$= 1,\ 0$:
9,	2	$1 + 9 \times \frac{1}{5}$	$= 2,\ 8$:
10,	2	$1 + 10 \times \frac{1}{5}$	$= 3,\ 0$:
15,	3	$1 + 15 \times \frac{1}{5} + \frac{15 \cdot 14}{1 \cdot 2} \times \frac{1}{25}$	$= 8,\ 2$:
16,	3	$1 + 16 \times \frac{1}{5} + \frac{16 \cdot 15}{1 \cdot 2} \times \frac{1}{25}$	$= 9,\ 0$:
21,	4	$1 + 21 \times \frac{1}{5} + \frac{21 \cdot 20}{1 \cdot 2} \times \frac{1}{25} + \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} \times \frac{1}{125}$	$= 24{,}24$:
22,	4	$1 + 22 \times \frac{1}{5} + \frac{22 \cdot 21}{1 \cdot 2} \times \frac{1}{25} + \frac{22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3} \times \frac{1}{125}$	$= 27,\ 0$:

That

$$\text{That is, when } m = \left\{\begin{array}{l}
\left\{\begin{matrix}3,\\ 4,\end{matrix}\right\} \text{then one term of the power is} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than} \\
\left\{\begin{matrix}9,\\ 10,\end{matrix}\right\} \text{then 2 terms are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than} \\
\left\{\begin{matrix}15,\\ 16,\end{matrix}\right\} \text{then 3 terms are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer.}\end{matrix}\right\} \text{than} \\
\left\{\begin{matrix}21,\\ 22.\end{matrix}\right\} \text{then 4 terms are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than}
\end{array}\right\} \overline{\tfrac{6}{5}}\Big|^{m} \times \tfrac{1}{2}$$

Th. when $m =$

$$\left\{\begin{matrix}6x+3,\\ 6x+4,\end{matrix}\right\} \text{then } x+1 \text{ terms of it are} \left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\} \text{than } \overline{\tfrac{6}{5}}\Big|^{m} \times \tfrac{1}{2}.$$

Therefore, when $m = 6x + 4$, there will be ſomething more than an equal chance of the required effects happening $x+1$ times.

Then ſince $m = 6x+4$; $\dfrac{m-4}{6} = x$:

And ſince $r = x+1$; $r-1 = x$:

Th. $r-1 = \dfrac{m-4}{6}$; And $r = \left(\dfrac{m-4}{6}+1=\right)\dfrac{m+2}{6}$.

That is, in a lottery, wherein there are five blanks to one prize, if m tickets are bought; then on an equality of chance, $\dfrac{m+2}{6}$ prizes may be expected.

COROL.

Since $r-1 = \dfrac{m-4}{6}$;

Th. $6r-6 = m-4$;

And $6r-2 = m$.

That is, if it be required to know, how many tickets ſhould, in ſuch a lotttery, be purchaſed, in order, on an equality of chance, to expect r prizes; the anſwer will be $6r-2$. Therefore, in order to have 1, 2, 3, 4, 5, &c. prizes, there ſhould be purchaſed 4, 10, 16, 22, 28, &c. tickets.

QUESTION LV.

In a lottery, wherein the number of blanks is to the number of prizes as b to a; how many tickets muſt be purchaſed to procure an equal chance for p or more prizes?

SOLUTION.

From a careful obſervation of the corollaries, to the preceeding five queſtions, it will appear, that the ſeries, which in each corollary expreſſes the number of tickets, which ought to be purchaſed, in order to procure an equality of chance for the having 1, 2, 3, 4, 5, &c. prizes, do ſeverally differ by 2, 3, 4, 5, 6; which, in each ſeparate queſtion; is the number of blanks more one; or (ſince there is ſuppoſed only one prize to a certain number of blanks) the number of chances which one ticket hath of being either blank or prize: Thus in corollary to queſt.

50	wherein the Lottery was ſuppoſed to have	1	blanks to one prizes; the N°. of tickets neceſſary to the expectation of 1, 2, 3, 4, 5, &c. prizes, was	1, 3, 5, 7, 9, &c;	which numbers differ by	2	the number of chances, which one ticket has of being either blank or prize.
51		2		2, 5, 8, 11, 14, &c;		3	
52		3		3, 7, 11, 15, 19, &c;		4	
53		4		4, 9, 14, 19, 24, &c;		5	
54		5		4, 10, 16, 22 28 &c;		6	

Therefore we may conclude, that the ſame thing will happen in all ſucceeding queſtions of this ſort; and conſequently, that if the firſt term of the ſeries can be obtained, then all the reſt may be found by the continual addition of $\overline{a+b}$.

Now the firſt term of this ſeries may be obtained by queſtion 49, where the number of tickets, which muſt be purchaſed, that the buyer may have an equal chance to have one prize, is $\frac{Log.\ 2.}{Log.\overline{a+b}-Log\ b}$. Therefore this quotient, if an integer, or the next greater integer, if a fraction, will be the firſt term of the ſeries: And if we call this quotient q, and put $a+b=s$; then, in order to have an equal chance, for 1, 2, 3, 4, 5, &c. prizes, we muſt purchaſe q, $q+s$, $q+2s$, $q+3s$, $q+4s$, &c. tickets; or univerſally, in order to have an equal chance for p prizes, we muſt purchaſe $q+\overline{p-1}\times s$ tickets.

QUESTION LVI.

Suppoſing the decrements of life to be equal; that is, ſuppoſing there be n perſons alive at any given age, and that one of them will die every year conſtantly, till they be all dead; it is required to find ($N=$) the preſent value of an annuity of 1 *l.* for a life of that age, allowing the purchaſer compound intereſt?

SOLUTION.

Let r be the amount of 1 *l.* in one year at compound intereſt; that is, the ſum of 1 *l.* and its intereſt for one year.

Then, becauſe at the end of one, two, three, &c. years, there will be, but $n-1$, $n-2$, $n-3$, &c. perſons alive, it will follow from corol. to queſt. 26, that the probability of the given life's ſurviving the firſt year will be $\frac{n-1}{n}$, that of the given life's ſurviving the ſecond year $\frac{n-2}{n}$, that of its ſurviving the third year $\frac{n-3}{n}$, &c. which expreſſions may be conſidered as the values of each payment of the annuity in the parts of 1 *l.*

But, becauſe the firſt, ſecond, third, &c. payments of this annuity are not to be made till the end of the firſt, ſecond, third, &c. years, it follows, from queſt. 144. part 2. vol. 1. that the preſent value of the firſt payment will be $\frac{n-1}{nr}$,

That of the ſecond payment $\frac{n-1}{nr^2}$,

That of the third payment $\frac{n-1}{nr^3}$, &c.

And therefore the value of the whole annuity will be $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}+\frac{n-4}{nr^4}\left(n\right)$; which ſeries may be divided into two other ſeries, *viz.*

$$\frac{n}{n}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}\,(n)=\frac{1}{1}\times\frac{1-p}{r-1}$$ By queſt. 15.

$$\frac{1}{n}\times\overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}}\,(n)=\frac{1}{n}\times\frac{\overline{1-p}\times r}{\overline{r-1}^{2}}-\frac{np}{r-1}$$ by 16.

Therefore *(N)* the value of the required annuity, will be

$$\frac{1-p}{r-1}-\frac{\overline{1-p}\times r}{n\times\overline{r-1}^{2}}+\left(\frac{np}{n\times r-1}=\right)\frac{p}{r-1}:$$

But $\frac{1-p}{r-1}+\frac{p}{r-1}=\frac{1}{r-1}$;

Therefore $N=\frac{1}{r-1}-\frac{\overline{1-p}\times r}{n\times\overline{r-1}^{2}}$,

Or $N=\frac{n\cdot\overline{r-1}}{n\times\overline{r-1}^{2}}-\frac{\overline{1-p}\times r}{n\times\overline{r-1}^{2}}$,

That is $N=\frac{nr-n-r+rp}{n\cdot\overline{r-1}^{2}}$;

Th. $N=\frac{\overline{n-1+p}\times r-n}{n\cdot\overline{r-1}^{2}}$;

where p is the preſent worth of 1 *l.* due at the end of n years.

This expreſſion produces the eaſieſt numerical proceſs, but that, below derived, will be hereafter wanted.

If $\frac{1}{r-1}=P$, and $\frac{r}{\overline{r-1}^{2}}=\mathcal{Q}$;

Then $\frac{1}{r-1}-\frac{\overline{1-p}\times r}{n\times\overline{r-1}^{2}}$ will become

$$P-\frac{\overline{1-p}}{n}\,\mathcal{Q};$$

That is $P - Q \times \overline{\frac{1}{n} - \frac{p}{n}}$,

Or $P + Q \times \overline{-\frac{1}{n} + \frac{1}{nr^n}}$.

EXAMPLE I.

If, at the age of fifty-four years, there are forty-three perſons alive; and that, from that time, the decrements of life are equal; what is the value of an annuity of 1 *l.* for a life of that age, allowing 4 *per Cent.* compound intereſt?

Here $n=43$ $p=0,185168$; And $r=1,04$

$$\begin{array}{rr} n=43 & n-1+p=42,185168 \\ r-1=,04 & r=\quad 1,04 \\ \hline 1,72 & 43,872574 \\ ,04 & -n=43 \\ \hline \end{array}$$

$,0688 = n \times \overline{r-1}^2$ $,0688)\ 0,872574\ (12,683 = N.$

EXAMPLE II.

If, at the age of fifty-four years, there are thirty-two perſons alive, (the decrements of life being ſuppoſed equal) what is the value of an annuity of 1 *l.* on that life, allowing 4 *per Cent.* compound intereſt?

Here $n=32$; $p=0,285058$; And $r=1,04$

$$\begin{array}{rr} n=32 & n-1+p=31,285058 \\ r-1=,04 & 1,04 \\ \hline 1,28 & 32,536460 \\ ,04 & -n=32,0 \\ \hline \end{array}$$

$,0512 = n \times \overline{r-1}^2$ $,0512)0,536460\ (10,478 = N.$

EXAMPLE III.

Suppoſing that, at the age of 66 years, there are 20 perſons alive, what will the value of an annuity of 1 *l.* on that life, be worth, allowing intereſt, &c. as above?

Here $n = 20$; $p = 0{,}456387$; And $r = 1{,}04$

$$\begin{array}{rr} n = 20 & n-1+p = 19{,}456387 \\ r-1 = {,}04 & 1{,}04 \\ \hline {,}80 & 20{,}234642 \\ {,}04 & -n\ 20{,}0 \\ \hline {,}0320 & {,}0320)\ 0{,}234642\ (7{,}333 = N. \end{array}$$

SCHOLIUM. The reader will obſerve, that, in theſe three examples, the number of perſons ſuppoſed to be alive at the ſeveral ages of 43, 54 and 66, are 43, 32 and 20; Whence it will appear, that in either caſe (if the decrements of life be equal) all the lives will be extinct at 86 (for 43+43; 54+32; and 66+20=86) which age, the juſtly celebrated Mr. *de Moivre* has aſſumed as the utmoſt probable extent of life. His words are theſe:

“ Another thing was neceſſary to my calculation,
“ which was, to ſuppoſe the extent of life confined to a
“ certain period of time, which I ſuppoſed to be at 86:
“ What induced me to aſſume that ſuppoſition was, 1*ſt.*
“ That Dr. *Halley* terminates his table of obſervations at
“ the 84th year; for although out of 1000 children of
“ one year of age, there are twenty, who, according to
“ Dr. *Halley*'s tables, attain to the age of 84 years, yet
“ that number is inconſiderable, and would ſtill have
“ been reduced, if the obſervations had been carried two
“ years farther. 2*d.* It appears from the tables of

“ *Graunt*, who printed the firſt edition of his book about “ 80 years ago, that out of 100 new born children, there “ remained not one after 86 years; this was deduced “ from the obſervations of ſeveral years, both in the “ city, and in the country, at the time when, the city “ being leſs populous, there was a greater facility of “ coming at the truth, than at preſent. 3*d*. I was farther “ confirmed in my hypotheſis, by tables of obſervations “ made, in *Switzerland*, about the beginning of “ this century, wherein the limit of life is placed at 86: “ As for what is alledged, that by ſome obſervations, of “ late years, it appears that life is carried to 90, 95, “ and to 100 years; I am no more moved by it, than “ by the examples of *Parr* or *Jenkins*, the firſt of which “ lived 152 years, and the other 167.”

In conformity, alſo, to that excellent gentleman, the number of perſons ſuppoſed to be alive at any age, being (as above obſerved) the differences between the given age and 86, ſhall be hereafter called the complement of life. But if the reader ſhall, for any good reaſon, think that 86 ought not to be taken for the outmoſt extent of life, and can pitch on a truer number, then, in all the ſubſequent rules, ſuch number muſt be ſubſtituted in the room of 86.

As the above rule for finding the value of a ſingle life, may frequently be wanted in practice, it is thought expedient to annex it, in words at length, as follows:

The rule to find the preſent value of an annuity of 1 l. *which is to continue during the life of a perſon of a given age, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal:*

Let the number of years, which the perſon wants of eighty-ſix, be called the complement of life; and let the ſum of one pound, and its Intereſt for one year, be called the rate.

Seek, in the tables, for the preſent worth of one pound, due at the end of the complement of life; to which add the complement of life leſs one.

Multiply the above ſum by the rate, and from the product take the complement of life; reſerving the remainder for a dividend.

Multiply the intereſt of one pound for one year by itſelf, and that product by the complement of life, for a diviſor.

Then, the quotient of that diviſion, will be the value of the annuity required.

EXAMPLE.

What is the value of an annuity of 1 *l.* for the life of a perſon of ten years of age, allowing compound intereſt at 4 *per Cent.*

Here (86—10=) 76 is the complement of life.
And (1+,04=) 1,04 is the rate.

The preſent worth of 1 *l.* due at the end of 76 years	0,05075
The complement of life leſs one is	75,00000
Their ſum is	75,05075
Which being multiplied by the rate	1,04
Will produce	78,05278
From which ſubtracting the complement of life	76,00000
There will remain for a dividend	2,05278

If the intereſt of 1 *l.* *viz.* 0,04 be multiplied by itſelf, the product will be 0,0016, which multiplied by 76 will produce 0, 1216 for a diviſor.

And if 2,05278 be divided by 0,1216 the quotient will be 16,8814 the value of the annuity required.

N. B. The value of an annuity of 1*l.* being multiplied by the yearly income of any other annuity, for the ſame life or lives, will give the value thereof: That is, the anſwers found to this, and the ſubſequent queſtions, may be conſidered as the number of years purchaſe, which ſuch annuities, as are deſcribed in the ſeveral queſtions, are worth.

This computation will be rendered ſomewhat eaſier, by uſing the table in page 77. Thus,

Take the preſent worth of one pound, due at the end of the complement of life, from unity; multiply the remainder by the number, which (in the above mentioned table) ſtands on a line, with the rate of intereſt, under the letter Q; *and divide this product by the complement of life.*

Subtract the above-found quotient from the number, which (on the ſame line of the ſame table) ſtands under the letter P; *and the remainder will be the value required.*

Thus in the above example,
The tabular number under *Q* is 650,
That under *P* is 25;

And if (1—0,05075=) 0,94925 be multiplied by 650, the product will be 617,013; which being divided by 76, will quote 8, 1186.

And if 8, 1186 be ſubtracted from 25, the remainder (*viz.* 16,8814) will be the value of the annuity, as before.

QUES-

QUESTION LVII.

The decrements of life being equal, ſuppoſe A, the complement of whoſe life is *m*, is to pay to B, or his aſſigns, an annuity of 1 *l.* *per Annum*, for *n* years certain, if A ſhould live ſo long; what is the preſent value of B's annuity?

Since every payment of B's annuity, depends upon the continuance of A's life, it will follow (by reaſoning as in queſt. 56) that it will be worth *n* terms of the ſeries $\frac{m-1}{mr}+\frac{m-2}{mr^2}+\frac{m-3}{mr^3}+\frac{m-4}{mr^4}$, &c. which may be divided into the two following ſeries, *viz.*

$$\frac{m}{m}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}\;(n)}=\overline{1-\frac{1}{r^n}}\times\frac{1}{r-1};$$ And

$$-\frac{1}{m}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}(n)}=\overline{1-\frac{1}{r^n}}\times\frac{r}{m\times\overline{r-1}^{\,2}}-\frac{1}{r^n}\times\frac{n}{m\times\overline{r-1}}$$

Thereſore the value of the required annuity will be,

$$\overline{1-\frac{1}{r^n}}\times\frac{1}{r-1}+\frac{n}{mr^n}\times\frac{1}{r-1}-\overline{1-\frac{1}{r^n}}\times\frac{r}{m\times r-1^{2}}:$$

Now $\frac{1}{1}-\frac{n}{m}=\frac{m-n}{m}$; Th. it will become $\overline{1-\frac{m-n}{m\times r^n}}\times\frac{1}{r-1}-\overline{1-\frac{1}{r^n}}\times\frac{r}{m\times\overline{r-1}^{\,2}}$.

E X A M P L E.

Suppoſe A, whoſe age is 43, is to pay to B an annuity of 1 *l.* for 32 years certain, if he (that is A) ſhould live ſo long: What is the preſent worth of B's annuity, allowing 4 *per Cent.* compound intereſt?

Here $m=(86-43=)43$; $n=32$; $\dfrac{1}{r^n}=0,285058$; & $r=1,04$

$$\begin{array}{rl} m= & 43 \\ r-1= & \underline{,04} \\ & 1,72 \\ & \underline{,04} \\ & ,0688 \end{array} \qquad \begin{array}{l} m=43; \\ \underline{n=32} \\ m-n=11 \end{array} \qquad \begin{array}{rl} m-n= & \overline{\quad 11 \quad} \\ m=43) & \underline{3,135638}(,072922 \end{array}$$

$$\begin{array}{r} 1,0 \\ \underline{-0,072922} \\ 1-\dfrac{m-n}{mr^n}=0,927078 \end{array} \qquad \begin{array}{r} 1 \\ \underline{-0.285058} \\ 1-\dfrac{1}{r^n}=0,714942 \end{array}$$

$$\begin{array}{r} 25=\dfrac{1}{r-1} \\ \overline{23,17695} \\ \underline{10,807} \\ 12,370 \end{array} \qquad \begin{array}{r} 1,04 \\ 0,688)\overline{0,743540}(10,807 \end{array}$$

12,370 *The Anſwer.*

QUESTION LVIII.

It is required to find the value of an annuity of 1 *l.* to continue *s* years, if a perſon of a given age ſhall live ſo long; allowing compound intereſt at a given rate, and ſuppoſing that (by a table of obſervations deduced from the bills of mortality of the place where the annuitant reſides) it ſhould appear, that the number of perſons living at the beginning, and end, of that period of time, (during which the annuity is to continue) are proportional to the numbers *a* and *b*, and that the decrements of life, are equal?

SOLU-

SOLUTION.

Since the decrements of life are equal, during the continuance of the propoſed annuity, 'tis evident that a and b muſt be the extreme terms of $\overline{s+1}$ numbers in arithmetical progreſſion; and therefore, the common difference of the numbers in that progreſſion will be $\frac{a-b}{s}$ by queſt. 6. part 2. vol. I.) That is, $a-\frac{a-b}{s}$, $a-\frac{2a-2b}{s}$, $a-\frac{3a-3b}{s}$, &c.

Or $\frac{sa-a+b}{s}$, $\frac{sa-2a+2b}{s}$, $\frac{sa-3a+3b}{s}$, &c.

will repreſent the numbers themſelves; and, if we argue as in queſt. 56, it will appear that the probabilities of the continuance of the given life for 1, 2, 3, &c. years, will be $\frac{sa-a+b}{sa}$, $\frac{sa-2a+2b}{sa}$, $\frac{sa-2a+2b}{sa}$, &c. the preſent worths of which probabilities, being added together, give $\frac{sa-a+b}{sar}+\frac{sa-2a+2b}{sar^2}+\frac{sa-3a+3b}{sar^3}\ (s)$ for the value of the annuity required.

Now the above may be divided into the two following ſeries, *viz.*

$$\frac{sa}{sa}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}\left(s\right)}=\frac{1}{sa}\times\overline{\frac{sa}{r-1}-\frac{sa}{\overline{r-1}\times r^s}};$$

$$-\frac{a-b}{sa}\times\overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}\left(s\right)}=$$

$$\frac{1}{sa}\times\begin{cases}-\frac{r\times\overline{a-b}}{\overline{r-1}^2}+\frac{s\times\overline{a-b}}{\overline{r-1}\times r^s}\\+\frac{r\times\overline{a-b}}{\overline{r-1}^2\times r^s}.\end{cases}$$

And therefore, the value of the annuity will be

$$\frac{1}{sa}\times\overline{\frac{sa}{r-1}-\frac{r\times\overline{a-b}}{\overline{r-1}^{2}}-\frac{sb}{\overline{r-1}\times r^{s}}+\frac{r\times\overline{a-b}}{\overline{r-1}^{2}\times r^{s}}}.$$

QUESTION LIX.

It is required to find the value of an annuity to continue $s+t$ years, if a perſon of a given age ſhall live ſo long; allowing compound intereſt at a given rate, and ſuppoſing that (by a table of obſervations deduced from the bills of mortality of the place where the annuitant reſides) it ſhould appear that the number of perſons living, at the beginning, at the end of s years, and at the end of $s+t$ years, are reſpectively proportional to the numbers a, b, and c; and that the decrements of life, for the ſeparate intervals of s and t years, are equal; and laſtly, that α is the difference between the common differences, of the two arithmetical progreſſions, of which a and b, and b and c, are the extreme terms. That is $\alpha=\frac{a-b}{s}-\frac{b-c}{t}$?

SOLUTION.

The value of the annuity for the firſt s years will be

$$\frac{1}{sa}\times\overline{\frac{sa}{r-1}-\frac{r\times\overline{a-b}}{\overline{r-1}^{2}}-\frac{sb}{\overline{r-1}\times r^{s}}+\frac{r\times\overline{a-b}}{\overline{r-1}^{2}\times r^{s}}},\text{ Or}$$

$$\frac{1}{r-1}-\frac{r}{\overline{r-1}^{2}}\times\frac{a-b}{sa}-\frac{1}{\overline{r-1}\times r^{s}}\times\frac{b}{a}+\frac{r}{\overline{r-1}^{2}\times rsa}\times\frac{a-b}{s}$$

by queſt. 58. and the probabilities of the given life's continuing $s+1$, $s+2$, $s+3$, &c. years will (by arguing as before) be $\frac{tb-b+c}{ta}$, $\frac{tb-2b+2c}{ta}$, $\frac{tb-3b+3c}{ta}$, &c. whence the value of the annuity for the laſt t years will be repreſented by the ſeries $\frac{tb-b+c}{tar^{s+1}}+\frac{tb-2b+2c}{tar^{s+2}}+\frac{tb-3b+3c}{tar^{s+3}}(t)$, which being ſummed in the ſame manner as the laſt will amount to

$$\frac{1}{tar^{s}}\times\overline{\frac{tb}{\overline{r-1}}-\frac{r\times\overline{b-c}}{\overline{r-1}^{2}}-\frac{tc}{\overline{r-1}\times r^{t}}+\frac{r\times\overline{b-c}}{\overline{r-1}^{2}\times r^{t}}},\ \text{Or}$$

$$\frac{1}{\overline{r-1}\times r^{s}}\times\frac{b}{a}-\frac{r}{\overline{r-1}^{2}\times r^{s}a}\times\frac{b-c}{t}-\frac{c}{\overline{r-1}\times ar^{s+t}}+\frac{r\times\overline{b-c}}{\overline{r-1}^{2}\times tar^{s+t}}$$

The ſum of which two values will be the value of the whole annuity: Now it is obſervable, that the third term of the value of the annuity for the firſt s years is the ſame with the firſt term of the value for the laſt t years, but with a contrary ſign, therefore thoſe terms will vaniſh out of the ſum; again the fourth term of the former hath the ſame common factor with the ſecond term of the latter $\left(viz.\ \frac{r}{\overline{r-1}^{2}\times r^{s}a}\right)$, thoſe terms have contrary ſigns, and the quantities multiplied into that factor, are $\frac{a-b}{s}$ and $\frac{b-c}{t}$, which have a given difference, *viz.* a, therefore thoſe two terms may be expreſſed by $\frac{ra}{\overline{r-1}^{2}\times r^{s}a}$; And the value of the whole annuity will be

$\frac{1}{r-1}$

$$\frac{1}{r-1}-\frac{r}{\overline{r-1}^{2}}\times\frac{a-b}{sa}+\frac{r\alpha}{\overline{r-1}^{2}\times r^{s}a}-\frac{c}{\overline{r-1}\times ar^{s+t}}$$

$$\left(\frac{r\times\overline{b-c}}{\overline{r-1}^{2}\times tar^{s+t}}\right.$$

QUESTION LX.

Let it be required to find the value of an annuity to continue $s+t+v$ years, if a perſon of a given age ſhall live ſo long; allowing compound intereſt at a given rate, and ſuppoſing that (by a table of obſervations deduced from the bills of mortality of the place where the annuitant reſides) it ſhould appear that the number of perſons living at the beginning, at the end of s years, at the end of $s+t$ years, and at the end of $s+t+v$ years, are reſpectively proportional to a, b, c, and d; that the decrements of life are ſeverally equal during the reſpective intervals of s, t, and v years, and laſtly that $\frac{a-b}{s}-\frac{b-c}{t}=\alpha$; And $\frac{b-c}{t}-\frac{c-d}{v}=\beta$?

SOLUTION.

The value of the annuity for the firſt $s+t$ years being found in the laſt queſtion, it remains only to find its value for the laſt v years: Now the probabilities of the given life's continuing $s+t+1$, $s+t+2$, $s+t+3$, &c. years are ſeverally

$\frac{vc-c+d}{va}$, $\frac{vc-2c+2d}{va}$, $\frac{vc-3c+3d}{va}$, &c. and therefore the value of the annuity, for the laſt interval of v years will be $\frac{vc-c+d}{var^{s+t+1}}+\frac{vc-2c+2d}{var^{s+t+2}}+\frac{vc-3c+3d}{var^{s+t+3}}$, (v) which, being ſummed in the ſame manner as the former, will become,

$$\frac{1}{var^{s+t}}\times\overline{\frac{vc}{\overline{r-1}}-\frac{r\times\overline{c-d}}{\overline{r-1}^{2}}-\frac{vd}{\overline{r-1}\times r^{v}}+\frac{r\times\overline{c-d}}{\overline{r-1}^{2}\times r^{v}}}\text{, Or}$$

$$\frac{1}{\overline{r-1}\times r^{s+t}}\times\frac{c}{a}-\frac{r}{\overline{r-1}^{2}\times ar^{s+t}}\times\frac{c-d}{v}-\frac{d}{\overline{r-1}\times ar^{s+t+v}}$$

$$\Big(+\frac{r\times\overline{c-d}}{\overline{r-1}^{2}\times var^{s+t+v}}.$$

And if we proceed to add the two firſt terms of this expreſſion, and the two laſt terms of the reſult of the laſt queſtion together, in the ſame manner as before, the ſum will be $\frac{r\beta}{\overline{r-1}^{2}\times ar^{s+t}}$; and conſequently the value of the whole annuity will be

$$\frac{1}{r-1}-\frac{r}{\overline{r-1}^{2}}\times\frac{a-b}{sa}+\frac{r\alpha}{\overline{r-1}^{2}\times a\times r^{s}}+\frac{r\beta}{\overline{r-1}^{2}\times ar^{s+t}}-$$

$$\Big(\frac{d}{\overline{r-1}\times ar^{s+t+v}}+\frac{r\times\overline{c-d}}{\overline{r-1}^{2}\times var^{s+t+v}}.$$

COROL. I.

The manner of continuing this proceſs is ſo evident, that it ſeems quite needleſs to purſue it farther, we ſhall therefore proceed to a general expreſſion thereof.

Now the above will, by reduction, become

$$\frac{1}{\overline{r-1}^2 \times a} \times \overline{\frac{\overline{r-1} \times a}{1} - \frac{\overline{a-b} \times r}{s} + \frac{\alpha}{r^s} + \frac{\beta}{r^{s+t}} - \frac{\overline{r-1} \times d}{r^{s+t+v}} + \frac{\overline{c-d} \times r}{vr^{s+t+v}}},$$

Or (putting $P = \frac{1}{r-1}$)

$$\frac{PP}{a} \times \overline{\frac{a}{P} - \frac{\overline{a-b} \times r}{s} + \frac{\alpha}{r^s} + \frac{\beta}{r^{s+t}} + \frac{d}{Pr^{s+t+v}} + \frac{\overline{c-d} \times r}{vr^{s+t+v}}}$$

If therefore from any table of obſervations (deduced from bills of mortality) it appears, that the numbers which are proportional to the living at the end of each year, do for s years decreaſe in an arithmetical progreſſion; for the next t years, in another arithmetical progreſſion; for the next v years in a third; and for the next w years in a fourth; and ſo on to the laſt z years wanted; and if the ſaid tabular numbers, at the beginning of each ſuch period be, ſeverally a, b, c, &c. to g; h, being the tabular number at the end of the period z; then will the common differences of thoſe arithmetical progreſſions be

$\frac{a-b}{s}$, $\frac{b-c}{t}$, $\frac{c-d}{v}$, $\frac{d-e}{w}$, &c. to $\frac{g-h}{z}$: And if, for the ſeveral differences of thoſe differences, we write α, β, γ, &c. to n; *viz.* $\frac{a-b}{s} - \frac{b-c}{t} = \alpha$; $\frac{b-c}{t} - \frac{c-d}{v} = \beta$ &c. Then, an annuity to continue $s+t+v$, &c. $+ z$ years, (the number of which intervals is m) if a life of that age, (which in the table of obſervations correſponds to the number of living a) ſhall continue ſo long, will be worth

PP

$$\frac{pP}{a}\times\left\{\begin{array}{l}\frac{a}{P}-\frac{\overline{a-b}\times r}{s}-\frac{b}{Pr^{s+t\,\&c.+z}}+\frac{\overline{g-b}\times r}{zr^{s+t\,\&c+z}}\\ +\frac{\alpha}{r^{s}}+\frac{\beta}{r^{s+t}}+\frac{\gamma}{r^{s+t+v}}+\frac{\delta}{r^{s+t+v+w}}(m-1):\end{array}\right.$$

Where α, β, γ, &c. will be found, moſt commonly, to denote +1 or −1, and their factors, $\frac{1}{r^{s}}$, $\frac{1}{r^{s+t}}$ &c. are the numbers, which in a table of the preſent worths of 1 *l* at the given rate, ſtand againſt the numbers, s, $s+t$, &c.

QUESTION LXI.

It is required to find the value of an annuity to continue during a life of a given age, ſuppoſing, as before, that by the table of obſervations, deduced from the bills of mortality of the place where the annuitant reſides, it ſhould appear, that the numbers, proportional to the perſons living of the ſucceeding ages may be divided into ſeveral ſuch arithmetical progreſſions as are above deſcribed, and allowing compound intereſt at a given rate?

SOLUTION.

If the ſame ſymbols be retained, as in the laſt queſtion; Then, ſince the value of an annuity for the whole life of that age is required, b will neceſſarily be inconſiderable,

ſiderable, and conſequently, $\frac{g-b}{z}$ will equal $\frac{g}{z}$, which cannot greatly differ from unity; whence ſuch an annuity may be expreſſed by

$$\frac{PP}{a}\times\overline{\frac{a}{P}-\frac{\overline{a-b}\times r}{s}+\frac{\alpha}{r^{s}}+\frac{\beta}{r^{s+t}}+\frac{\gamma}{r^{s+t+v}}}\,(m).$$

But $\frac{a-b}{s}$ is the common difference of the firſt arithmetical progreſſion, that is, the difference between the two firſt tabular numbers, a and the next leſſer, which difference may be ſeen by inſpection on the table, and may be denoted by D: and then the annuity will be worth

$$\frac{PP}{a}\times\overline{\frac{a}{P}-Dr+\frac{\alpha}{r^{s}}+\frac{\beta}{r^{s+t}}+\frac{\gamma}{r^{s+t+v}}}\,(m).$$

COROL. I.

Hence if (by any table of ſuch obſervations) it ſhould appear, that, in the numbers proportional to the living, the ſeveral arithmetical progreſſions ſhould each conſiſt of an equal number of terms; and that their common differences ſhould, alſo, be in arithmetical progreſſion; then the annuity would be expreſſed by

$$\frac{PP}{a}\times\overline{\frac{a}{P}-Dr+\frac{\alpha}{r^{s}}+\frac{\alpha}{r^{2s}}+\frac{\alpha}{r^{3s}}}\,(m).$$

But $\frac{\alpha}{r^{s}}+\frac{\alpha}{r^{2s}}+\frac{\alpha}{r^{3s}}$, &c. is a geometrical progreſſion, whoſe greateſt term is $\frac{\alpha}{r^s}$, ratio r^s, and number of terms m; therefore the ſum thereof will (by queſt. 95. part 2. vol. I.) be $\frac{\overline{r^{sm}-1}\times\alpha}{r^s\times\overline{r^s-1}\times r^{sm-1}}$, and conſequently the annuity will be worth

$$\frac{PP}{a}\times\overline{\frac{a}{P}-Dr+\frac{\overline{r^{sm}-1}\times\alpha}{r^s\times\overline{r^s-1}\times r^{sm-1}}}.$$

COROL. II.

If the numbers, proportional to the living, ſhould be a ſeries whoſe ſecond differences are equal; then will the value of the annuity be repreſented by

$$\frac{PP}{a}\times\overline{\frac{a}{P}-Dr+\frac{\alpha}{r}+\frac{\alpha}{r^2}+\frac{\alpha}{r^3}\,(m)}.$$

Where m would repreſent the ſame thing as was (in queſt. 56) called the complement of life, and the annuity will (by queſt. 15) become

$$\frac{PP}{a}\times\overline{\frac{a}{P}-Dr+\frac{\alpha}{r-1}-\frac{\alpha}{\overline{r-1}\times r^m}}.$$

But as the two latter caſes will probably ſeldom occur, it may be ſufficient, here, to give a rule in words at length for the former, and more general caſe; in order to which, it may be convenient to give ſome directions, concerning the diſpoſition of a table of obſervations.

Let then the table of obſervations conſiſt of four columns.

In that column, which is next to the left hand, inſert the numbers 1, 2, 3, *&c. to repreſent the years of life, from infancy to the extremity of old age.*

In the ſecond column, inſert thoſe numbers which (from the bills of mortality of the place) appear to be proportional to the number of perſons living, of each particular age, contained in the firſt column; the firſt number, therefore, of this column will be the greateſt, and every preceding number will be greater than the ſucceeding, untill the laſt, which will be the leaſt.

In the third column, let the differences, between the numbers of the ſecond column, be placed, in ſuch a manner, that the difference between any number, and the next leſſer may ſtand on a line with the greater number; thoſe numbers will be proportional to the number of perſons, which die between the age, againſt which they ſtand, and the next following age, mentioned in the table: When the numbers, which compoſe this column, continue the ſame in one, two, three, four, &c. ſucceſſive ages, they conſtitute ſuch an interval of years, as has been above repreſented by the ſymbols s, t, v, *&c. Let therefore every ſuch interval be (by ſomething remarkable,* namely, *a rule, or larger ſpace, than common) diſtinguiſhed from the preceding.*

In the fourth column, let there be placed, on the firſt line of each interval, the difference between the laſt number in the third column of the former interval, and the firſt number in the ſame column of the preſent interval; placing before it the ſign +, *if the number in the former interval exceeds that in the preſent; and the ſign* —, *if the number in the former interval be leſs than that in the preſent*

ſent: Theſe numbers, with their ſigns, are above denoted by α, β, γ, *&c.*

The following TABLE *of* OBSERVATIONS, *deduced (by the very ingenious Mr.* Tho. Simpſon) *from the Bills of Mortality of* LONDON, *being put into the above-preſcribed Form, may ſerve as an Example.*

Ages born	Perſons living	D	α
	1280	410	
1	870	170	+240
2	700	65	+105
3	635	35	+ 30
4	600	20	+ 15
5	580	16	+ 4
6	564	13	+ 3
7	551	10	+ 3
8	541	9	+ 1
9	532	8	+ 1
10	524	7	+ 1
11	517	7	
12	510	6	+ 1
13	504	6	
14	498	6	
15	492	6	
16	486	6	
17	480	6	
18	474	6	
19	468	6	
20	462	7	− 1
21	455	7	
22	448	7	
23	441	7	
24	434	8	− 1
25	426	8	
26	418	8	
27	410	8	
28	402	8	
29	394	9	− 1
30	385	9	
31	376	9	
32	367	9	
33	358	9	
34	349	9	
35	340	9	
36	331	9	
37	322	9	
38	313	9	
39	304	10	− 1
40	294	10	
41	284	10	
42	274	10	
43	264	9	+ 1
44	255	9	
45	246	9	
46	237	9	
47	228	8	+ 1
48	220	8	
49	212	8	

Ages

Ages born	Perſons living	D	a
50	204	8	
51	196	8	
52	188	8	
53	180	8	
54	172	7	+ 1
55	165	7	
56	158	7	
57	151	7	
58	144	7	
59	137	7	
60	130	7	
61	123	6	+ 1
62	117	6	
63	111	6	
64	105	6	
65	99	6	
66	93	6	
67	87	6	
68	81	6	
69	75	6	
70	69	5	+ 1
71	64	5	
72	59	5	
73	54	5	
74	49	4	+ 1
75	45	4	
76	41	3	+ 1
77	38	3	
78	35	3	
79	32	3	
80	29	3	
81	26	3	
82	23	3	
83	20	3	
84	17	3	
85	14	2	+ 1
86	12	2	
87	10	2	
88	8	2	
89	6	1	+ 1
90	5	1	
91	4	1	
92	3	1	
93	2	1	
94	1	1	

The author, mentioned in the title of this table, has carried his numbers no farther than 80 years; but another noted author, who very nearly agrees with the former in every age above 25, has continued his numbers lower in the above manner.

The following TABLE *of* OBSERVATIONS, *deduced (by the juſtly celebrated Dr.* HALLEY) *from the Bills of Mortality of* BRESLAW, *is as a farther Example, inſerted in the Form above recommended.*

Ages	Perſons living	D	a
1	1000	145	
2	855	57	+ 88
3	798	38	+ 19
4	760	28	+ 10
5	732	22	+ 6
6	710	18	+ 4
7	692	12	+ 6
8	680	10	+ 2
9	670	9	+ 1
10	661	8	+ 1
11	653	7	+ 1
12	646	6	+ 1
13	640	6	
14	634	6	
15	628	6	
16	622	6	
17	616	6	
18	610	6	
19	604	6	
20	598	6	
21	592	6	
22	586	7	− 1
23	579	6	+ 1
24	573	6	
25	567	7	− 1
26	560	7	
27	553	7	
28	546	7	
29	539	8	− 1
30	531	8	
31	523	8	
32	515	8	
33	507	8	
34	499	9	− 1
35	490	9	
36	481	9	
37	472	9	
38	463	9	
39	454	9	
40	445	9	
41	436	9	
42	427	10	− 1
43	417	10	
44	407	10	
45	397	10	
46	387	10	
47	377	10	
48	367	10	
49	357	11	− 1
50	346	11	
51	335	11	
52	324	11	
53	313	11	
54	302	10	+ 1

Ages

Ages	Perſons living	D	α
55	292	10	
56	282	10	
57	272	10	
58	262	10	
59	252	10	
60	242	10	
61	232	10	
62	222	10	
63	212	10	
64	202	10	
65	192	10	
66	182	10	
67	172	10	
68	162	10	
69	152	10	
70	142	11	− 1
71	131	11	
72	120	11	
73	109	11	
74	98	10	+ 1
75	88	10	
76	78	10	
77	68	10	
78	58	9	+ 1
79	49	8	+ 1
80	41	7	+ 1
81	34	6	+ 1
82	28	5	+ 1
83	23	4	+ 1
84	19	4	
85	15	4	
86	11	3	+ 1
87	8	3	
88	5	2	+ 1
89	3	2	
90	1	1	+ 1

The numbers, proportional to the perſons living, in this table were copied from Dr. *Halley*'s table, as inſerted at the end of Mr. *De Moivre*'s doctrine of chances, which is carried no farther than 84 years; againſt which the number 20 is placed, where I have here inſerted 19; which alteration I made, becauſe, otherwiſe the number in the fourth column, even with 83, would have been 2, contrary to the general law obſerved, in all other inſtances, both in this and the *London* table, after the age of 8 years: The numbers, placed againſt the following years, were inſerted in that manner, which I conceived to be moſt conformable to that general law; but [illegible] have erred therein, the value of a life will be but little affected thereby.

The RULE,

To find the preſent value of an annuity of 1 l. to continue during a ſingle life of a given age, allowing compound intereſt at a given rate, by the aſſiſtance of a table of obſervations (deduced from the bills of mortality of the place where the annuitant reſides) diſpoſed in the manner above deſcribed, and a table of the preſent worths of 1 l. ſterling, due at the end of any number of years to come.

From the firſt number of the left hand column of each interval, which follows the given age in the tables of obſervations, take the given age, and let the remainders be called the complements of each interval; alſo let 1 l. and its intereſt be called the rate.

As a title of diſtinction, between the two hereafter directed columns of figures, place the ſigns + and —.

Multiply ſeverally the preſent worths of 1 l due at the ends of thoſe numbers of years, which are expreſſed by the reſpective complements of each of the above-mentioned intervals, by the number which ſtands in the fourth column of the ſaid table, on the firſt line of each interval, placing the products under the ſigns +, or —, according to the ſigns prefix'd to the laſt mentioned numbers.

Multiply the number, which ſtands in the ſecond column of the table of obſervations, on a line with the given age, by the intereſt of 1 l. placing the product under the ſign +: Alſo multiply the laſt found product, again, by the intereſt of 1 l. reſerving the product for future uſe.

Multiply the number which ſtands in the third column of the table of obſervations, on a line with the given age, by the rate; and place this product under the ſign —.

Subtract the ſum of all the numbers, which ſtand under the ſign —, from the ſum of all thoſe which ſtand under *the ſign +; and divide the remainder by the product above reſerv'd; ſo ſhall the quotient be the value of the annuity required.*

EXAMPLE I.

Let the preſent value of an annuity of 1 *l.* on a life aged ten years, be computed (at 4 *per Cent.*) by the above given table of obſervations, deduced from the bills of mortality at *London*.

Here, the firſt numbers ſtanding in the left hand column of each interval, following 10 years, are

12, 20, 24, 29, 39, 43, 47, 54, 61, 70, 74, 76, 85, 89.

From each of which, if 10 the given age be taken, there will remain

2, 10, 14, 19, 29, 33, 37, 44, 51, 60, 64, 66, 75, 79,

the complements of the ſeveral intervals; and the numbers which ſtand in the fourth column, and firſt line of each interval will appear to be,

$$\begin{array}{cccccccccccccc} + & - & - & - & - & + & + & + & + & + & + & + & + & + \\ 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1. \end{array}$$

The preſent worths of one pound, due at the end of 2, 10, 14, *&c.* years, being multiplied by theſe units, and being properly ranged, will ſtand as below, Now, if 524 the number of perſons living at 10 years of age, be multiplied by 0,04, the intereſt of 1 *l.* it produces 20,96, which is placed under +: And if this be, again, multiplyed by 0,04, it produces 0,8384 the diviſor afterwards uſed.

Alſo 7 the number of perſons dying between the ages 10 and 11, being multiplied by (1+0,04=) 1,04 the rate, produces 7,28 which is placed under —.

years	+	years	—
2	0,9246	10	0,6756
33	0,2741	14	0,5775
37	0,2343	19	0,4746
44	0,1780	29	0,3207
51	0,1353		7,2800
60	0,0951		9,3284
64	0,0813		
66	0,0751		
75	0,0528		
79	0,0451		
	20,9600		

Sum + 23,0557
Sum — 9,3284

0,8384) 13,7273 (16,3732
the value of the annuity required.

EXAMPLE II.

Let the preſent worth of an annuity of 1 *l.* on a life aged ten years, be computed (at 4 *per Cent.)* by the above given table of obſervations, deduced from the bills of mortality at *Breſlaw?*

Here the firſt numbers of the left hand column of each interval, following 10 years, are

11, 12, 22, 23, 25, 29, 34, 42, 49, 54, 70, 74, 78, 79, 80,
(81, 82, 83, 86, 88, 90.

From each of which the given age 10, being ſubtracted, there will remain

1, 2, 12, 13, 15, 19, 24, 32, 39, 44, 60, 64, 68, 69, 70,
(71, 72, 73, 76, 78, 80,

the complements of each interval.

And the numbers which ſtand in the fourth column of each interval are

$\overset{+}{1}, \overset{+}{1}, \overset{-}{1}, \overset{+}{1}, \overset{-}{1}, \overset{-}{1}, \overset{-}{1}, \overset{-}{1}, \overset{-}{1}, \overset{+}{1}, \overset{-}{1}, \overset{+}{1}, \overset{+}{1}, \overset{+}{1}, \overset{+}{1}, \overset{+}{1}, \overset{+}{1}, \overset{+}{1},$
$(\overset{+}{1}, \overset{+}{1}, \overset{+}{1}.$

The preſent worths of 1 *l.* due at the end of 1, 2, 12, 13, *&c.* years, being properly ranged, will ſtand as below

years	+	years	—
1	0,9615	12	0,6246
2	0,9246	15	0,5553
13	0,6006	19	0,4746
44	0,1780	24	0,3901
64	0,0813	32	0,2851
68	0,0695	39	0,2166
69	0,0668	60	0,0951
70	0,0642		8,3200
71	0,0618		
72	0,0594		10,9614
73	0,0571		
76	0,0508		
78	0,0469		
80	0,0434		
	26,4400		

Sum + 29,7059
Sum — 10,9614

1,0576) 18,7445 (17,7237 the anſwer.

Now if 661, the living at 10 years, be multiplied by 0,04, the intereſt of 1 *l.* it produces 26,44, which is placed under +: And $(26{,}44 \times 0{,}04 =)$ 1,0576 is the diviſor afterwards uſed.

Alſo 8, the number dying between the ages of 10 and 11, being multiplied by $(1+0{,}04=)$ 1,04 the rate; produces 8,32, which is placed under —.

SCHO-

SCHOLIUM.

If the reſult of the two laſt examples (whereof the firſt gives 16,3732 *l.* for the value of an annuity of 1 *l.* for a life of 10 years, according to the *London* bills; and the latter gives 17,7237 *l.* for the value of an annuity on the ſame life, according to the *Breſlaw* bills) be compared with the value of the ſame annuity, ſuppoſing the decrements of life to be equal; which (by queſt. 56) was found to be 16, 8814; it appears that the reſult, according to that hypotheſis, is greater than the former, and leſſer than the latter; and conſequently, for general uſe, may be more eligible than either.

As the table of the preſent worths of 1*l.* due at the end of any number of years to come, is neceſſary to the ſolution of this and many other queſtions relating to annuities on lives; it is inſerted here, that the reader may not have the inconvenience of turning to another book, for the ſame.

And becauſe the values of ſingle lives, computed upon the ſuppoſition of the equality of the decrements of life, are alſo neceſſary to the approximation to the values of ſuch combined lives, a table containing them is alſo annexed.

A TABLE *of the preſent Values of One Pound.*

	3 *per Ct.*	3½ *per Ct.*	4 *per Ct.*	4½ *per Ct.*	5 *per Ct.*	6 *per Ct.*
1	,970874	,966184	,961538	,956938	,952381	,943396
2	,942596	,933511	,924556	,915730	,907029	,889996
3	,915142	,901943	,888996	,876297	,863838	,839619
4	,888487	,871442	,854804	,838561	,822702	,792094
5	,862609	,841973	,821927	,802451	,783526	,747258
6	,837484	,813501	,790315	.767896	,746215	,704961
7	,813092	,785991	,759918	,734828	,710681	,665057
8	,789409	,759412	,730690	,703185	,676839	,627412
9	.766417	,733731	,702587	,672904	,644609	.591898
10	.744094	,708919	,675564	,643928	.613913	,558395
11	,722421	,684946	,649581	,616199	.584679	:526788
12	,701380	,661783	,624597	.589664	.556837	,496969
13	,680951	,639404	,600574	.564272	.530321	,468839
14	,661118	,617782	,577475	539973	.505068	,442301
15	641862	,596891	,555265	.516720	,481017	,417265
16	,623167	,576706	,533908	,494469	,458112	,393646
17	,605016	,557204	,513373	,473176	,436297	,371364
18	,587395	,538361	,493628	,452800	,415521	,350344
19	.570286	,520156	,474642	,433302	,395734	,330513
20	.553676	,502566	,456387	414643	,376889	,311805
21	,537549	,485571	,438834	,396787	.358942	,294155
22	,521893	,469151	,421955	,379701	.341850	.277505
23	.506692	,453286	,405726	,363350	,325571	.261797
24	,491934	,437957	.390121	,347703	,310068	,246979
25	,477606	,423147	,375117	.332731	,295303	,232999
26	,463695	,408838	.360689	,318402	.281241	,219810
27	,450189	.395012	.346817	,304691	,267848	,207368
28	,437077	,381654	,333477	,291571	,255094	,195630
29	,424346	,368748	,320651	,279015	.242946	,184557
30	,411987	356278	,308319	,267000	,231377	,174110
31	.399987	344230	,296460	,255502	.220359	‘164255
32	,388337	.332590	,285058	,244500	,209866	,154957
33	,377026	,321343	,274094	,233971	,199873	,146186

A TABLE

A TABLE *of the present Values of One Pound.*

	3 *per Ct.*	3½ *per Ct.*	4 *per Ct.*	4½ *per Ct.*	5 *per Ct.*	6 *per Ct.*
34	,366045	,310476	,263552	,223896	,190355	,137912
35	,355383	,299977	,253415	,214254	,181290	,130105
36	,345032	,289833	,243669	,205028	,172657	,122741
37	,334983	,280032	,234297	,196199	,164436	,115793
38	,325226	,270562	,225285	,187750	,156605	,109239
39	,315754	,261413	,216621	,179665	,149148	,103056
40	,306557	,252572	,208289	,171929	,142046	,097222
41	,297628	,244031	,200278	,164525	,135282	,091719
42	,288959	.235779	,192575	,157440	,128840	,086527
43	,280543	.227806	,185168	,150663	,122704	,081630
44	,272372	,220102	,178046	,144173	,116864	,077009
45	,264439	,212659	,171198	,137964	,111297	,072650
46	,256737	,205468	,164614	,132023	,105997	,068538
47	,249259	,198520	,158283	,126338	,100949	,064658
48	,241999	,191806	,152195	,120898	,096142	,060998
49	,234950	,185320	,146341	,115692	,091564	,057546
50	,228107	,179053	,140713	,110710	,087204	,054288
51	,221463	,172998	,135301	,105942	,083051	,051215
52	,215013	,167148	,130097	,101380	,079096	,048316
53	,208750	,161496	,125093	,097014	,075330	,045582
54	,202670	,156035	,120282	,092837	,071743	,043001
55	,196767	,150758	,115656	,088839	,068326	,040567
56	,191036	,145660	,111207	,085013	,065073	,038271
57	.185472	,140734	,106930	,081353	,061974	,036105
58	,180070	,135975	,102817	,077849	,059023	,034061
59	,174825	,131377	,098863	,074497	,056212	,032133
60	,169733	,126934	,095060	,071289	,053536	,030314
61	,164789	,122642	,091404	,068219	,050986	,028598
62	,159990	118495	,087889	,065281	,048558	,026989
63	,155330	,114487	,084508	,062470	,046246	,025453
64	,150806	,110616	,081258	,059780	,044044	,024012
65	,146413	,106875	,078133	,057206	,041946	,022653
66	,142149	.103261	,075128	,054742	,039949	,021370

A TABLE

A TABLE *of the present Values of One Pound.*

	3 *per Ct.*	3½ *per Ct.*	4 *per Ct.*	4½ *per Ct.*	5 *per Ct.*	6 *per Ct.*
67	,138009	,099769	,072238	,052385	,038047	,020161
68	,133989	,096395	,069460	,050129	,036235	,019020
69	,130086	,093136	.066788	,047971	,034509	,017943
70	,126297	,089986	,064219	,045905	,032866	,016927
71	,122619	,086943	,061749	,043928	,031301	,015969
72	,119047	,084003	,059374	,042037	,029811	,015065
73	,115580	,081162	,057091	,040226	,028391	,014212
74	,112214	,078418	,054895	,038494	,027039	,013408
75	,108945	,075766	,052784	,036836	,025752	,012649
76	,105772	,073204	050754	,035250	,024525	,011933
77	.102691	,070728	,048801	,033732	,023357	,011258
78	,099700	,068336	,046924	,032280	,022245	,010620
79	,096796	,066026	,045120	,030890	,021186	,010019
80	,093977	,063793	,043384	,029559	,020177	,009452
81	,091240	,061636	,041716	,028287	,019216	,008917
82	,088582	,059551	,040111	,027068	,018301	,008412
83	,086002	,057538	,038569	,025903	,017430	,007936
84	,083497	,055592	,037085	,024787	,016600	,007487
85	,081065	,053712	,035659	,023720	,015809	,007063
86	,078704	,051896	,034287	,022699	,015056	,006663
87	,076412	,050141	,032968	,021721	,014339	,006286
88	,074186	,048445	,031700	,020786	,013657	,005930
89	,072027	,046807	,030481	,019891	,013006	,005595
90	,069928	,045224	,029309	,019034	,012387	,005278
91	,067891	,043695	,028182	,018215	,011797	,004979
92	,065914	,042217	,027098	,017430	,011235	,004697
93	,063994	,040789	,026055	,016680	,010700	,004432
94	,062130	,039410	,025053	,015961	.010191	,004181
95	,060320	,038077	,024090	,015274	.009705	,003944
96	,058563	,036790	,023163	,014616	,009243	,003721
97	,056858	,035546	,022272	,013987	,008803	,003510
98	,055202	,034344	,021416	,013385	,008384	,003312
99	,053594	,033182	,020592	,012808	,007985	,003124
110	,052033	,032060	,019800	,012257	,007604	,002957

A TABLE *of the preſent Values of an Annuity of One Pound, on a ſingle Life, ſuppoſing the Decrements of Life to be equal.*

Age	3 *perCt.*	3½ *perCt.*	4 *per Ct.*	4½ *perCt.*	5 *per Ct.*	6 *per Ct.*
8	19,736	18,160	16,791	15,595	14,544	12,790
9	19,868	18,269	16,882	15,672	14,607	12,839
10	19,868	18,269	16,882	15,672	14,607	12,839
11	19,736	18,160	16,791	15,595	14,544	12,790
12	19,604	18,049	16,698	15,517	14,480	12,741
13	19,469	17,937	16,604	15,437	14,412	12,691
14	19,331	17,823	16,508	15,356	14,342	12,639
15	19,192	17,707	16,410	15,273	14,271	12,586
16	19,050	17,588	16,311	15,189	14,197	12,532
17	18,905	17,467	16,209	15,102	14,123	12,476
18	18,759	17,344	16,105	15,015	14,047	12,419
19	18,610	17,220	15,999	14,923	13,970	12,361
20	18,458	17,093	15,891	14,831	13,891	12,301
21	18,305	16,963	15,781	14,737	13,810	12,239
22	18,148	16,830	15,669	14,641	13,727	12,177
23	17,990	16,696	15,554	14,543	13,642	12,112
24	17,827	16,559	15,437	14,442	13,555	12,045
25	17,664	16,419	15,318	14,340	13,466	11,978
26	17,497	16,277	15,197	14,235	13,375	11,908
27	17,327	16,133	15,073	14,128	13,282	11,837
28	17,154	15,985	14,946	14,018	13,186	11,763
29	16,979	15,835	14,816	13,905	13,088	11,688
30	16,800	15,682	14,684	13,791	12,988	11,610
31	16,620	15,526	14,549	13,673	12,885	11,530
32	16,436	15,367	14,411	13,553	12,780	11,449
33	16,248	15,204	14,270	13,430	12,673	11,365
34	16,057	15,039	14,126	13,304	12,562	11,278
35	15,864	14,871	13,979	13,175	12,449	11,189
36	15,666	14,699	13,829	13,044	12,333	11,098

A TABLE *of the preſent Values of an Annuity of One Pound, on a ſingle Life, ſuppoſing the Decrements of Life to be equal.*

Age	3 *per Ct.*	3½ *per Ct.*	4 *per Ct.*	4½ *per Ct.*	5 *per Ct.*	6 *per Ct.*
37	15,465	14,524	13,676	12,909	12,214	11,003
38	15,260	14,345	13,519	12,771	12,091	10,907
39	15,053	14,163	13.359	12,630	11,966	10,807
40	14,842	13,978	13 196	12,485	11,837	10,704
41	14 626	13,789	13,028	12,337	11,705	10,599
42	14,407	13,596	12,858	12,185	11,570	10,490
43	14.185	13,399	12,683	12,029	11,431	10,378
44	13.958	13,199	12,504	11,870	11,288	10,263
45	13,728	12,993	12,322	11,707	11,142	10,144
46	13,493	12,784	12,135	11,540	10,992	10,021
47	13,254	12,571	11,944	11,368	10,837	9,895
48	13,012	12,354	11,748	11,192	10,679	9.765
49	12,764	12,131	11,548	11,012	10,515	9.630
50	12,511	11,904	11,344	10,827	10,348	9,492
51	12,255	11,673	11,135	10,638	10,176	9,349
52	11,994	11,437	10,921	10,443	9,999	9,201
53	11,729	11,195	10,702	10,243	9,817	9 049
54	11,457	10,950	10,478	10,039	9,630	8,891
55	11,183	10,698	10,248	9,829	9,437	8,729
56	10,902	10,443	10,014	9,614	9,239	8,561
57	10,616	10,181	9,773	9,393	9 036	8,387
58	10;325	9.913	9,527	9,166	8,826	8,208
59	10,029	9,640	9,275	8,933	8,611	8,023
60	9,727	9,361	9,017	8,694	8,389	7,831
61	9,419	9,076	8,753	8,449	8,161	7.633
62	9,107	8,786	8,482	8,197	7,926	7,428
63	8,787	8,488	8,205	7,938	7,684	7,216
64	8,462	8,185	7,921	7,672	7,435	6,997
65	8,132	7,875	7,631	7,399	7,179	6,770

A TABLE

A TABLE *of the present Values of an Annuity of One Pound, on a single Life, supposing the Decrements of Life to be equal.*

Age	3*perCt.*	3½*perCt.*	4 *per Ct.*	4½*perCt.*	5 *per Ct.*	6 *perCt.*
66	7,794	7,558	7,333	7,119	6,915	6,535
67	7,450	7,234	7,027	6,831	6,643	6,292
68	7,099	6,902	6.714	6,534	6.362	6,040
69	6,743	6,565	6,394	6,230	6,073	5.779
70	6,378	6,219	6,065	5,918	5,775	5,508
71	6,008	5.865	5,728	5,596	5.468	5,228
72	5,631	5.505	5.383	5.265	5,152	4,937
73	5,246	5,136	5,029	4 926	4.826	4,636
74	4,854	4,759	4,666	4,576	4,489	4,324
75	4,453	4,373	4,293	4,217	4,143	4,000
76	4,046	3,978	3,912	3,847	3.784	3 664
77	3,632	3,575	3,520	3,407	3,415	3 315
78	3,207	3.163	3,111	3,076	3,034	2,953
79	2,776	2,741	2,707	2,673	2,641	2,578

QUESTION LXII.

Suppoſing the decrements of life to be in a conſtant ratio; that is, ſuppoſing the number of chances for its continuance in being one year, to be to the number of chances for its failing in that time, as a to b; the number of chances for its continuance the ſecond year, to the number of chances for its failing, as aa to bb, *&c.* it is required to find v the preſent value of 1 *l.* for that life, allowing the purchaſer compound intereſt?

SOLUTION.

Let r be the amount of 1 *l.* in one year. Then the probability of the given life's ſurviving the firſt year will be $\frac{a}{\overline{a+b}}$;

That of ſurviving the ſecond year $\frac{aa}{\overline{a+b}^2}$;

That of the third $\frac{aaa}{\overline{a+b}^3}$;

Which expreſſions, being taken as the values of each payment in the parts of 1 *l.* the ſum of their preſent values will be the preſent value of the annuity, *viz.*

$$\frac{a}{\overline{a+b}\times r}+\frac{aa}{\overline{a+b}^2\times r^2}+\frac{aaa}{\overline{a+b}^3\times r^3}+\frac{a^4}{\overline{a+b}^4\times r^4};\ \&c.$$

which, being a geometrical progreſſion, infinitely decreaſing, whoſe greateſt term is $\frac{a}{\overline{a+b}\times r}$, ratio $\frac{\overline{a+b}\times r}{a}$, and the ratio leſs unity $\left(\frac{\overline{a+b}\times r}{a} - 1 =\right) \frac{\overline{a+b}\times r-a}{a}$, the ſum thereof will be the quotient ariſing from the diviſion of $\frac{a}{\overline{a+b}\times r} \times \frac{\overline{a+b}\times r}{a}$ by $\frac{\overline{a+b}\times r-a}{a}$: But $\frac{a}{\overline{a+b}\times r} \times \frac{\overline{a+b}\times r}{a} = \frac{1}{1} = 1$; And 1 divided by $\frac{\overline{a+b}\times r-a}{a}$ will quote $\frac{a}{\overline{a+b}\times r-a}$ $(=N)$ the preſent value required, by queſt. 169. part 2. vol. I.

E X A M P L E.

Suppoſe a life, the number of chances for the continuance of which for one year, is to the number of chances for its failing in that time, as 1 to 0,03735, and that the decrements of life are in a conſtant ratio, what is its preſent value, allowing 4 *per Cent* compound intereſt?

Here $a=1$; $b=0{,}03735$; $a+b=1{,}03735$; & $r=1{,}04$:
Then $a+b=1{,}03735$,
$r=\ \ 1{,}04$.

$\overline{a+b}\times r=1{,}07884$
$\overline{a+b}\times r-a=0{,}07884$; And $N=\left(\frac{1}{,07884}=\right)$ 12,683.

QUESTION LXIII.

Suppoſing the decrements of life to be in a conſtant ratio: If the value of an annuity thereon be given, together with the rate of compound intereſt, it is required to find the probability of that life's continuing for one year.

SOLUTION.

If the ſymbols be retained as in the laſt queſtion; Then $\frac{a}{\overline{a+b}\times r-a} = N$; by the reſult thereof; and it is required to find the value of $\frac{a}{a+b}$, in the terms of N, and r.

Now $a = (N\times\overline{a+b\times r-a} =)\ \overline{a+b}\times rN-aN$;

And $a+aN = \overline{a+b}\times rN$;

Th. $\frac{a+aN}{a+b} = rN$

And $\frac{a}{a+b} = \frac{rN}{1+N}$.

EXAMPLE I.

If $N=12,683$ and $r=1,04$ as before, then, $rN = 13,19032$ And $\left(\frac{13,19032}{13,683} =\right) 0,963993 = \frac{a}{a+b}$

EXAMPLE II.

If $N=10,478$, and $r=1,04$:

Then $rN = 10,89712$; and $\frac{a}{a+b} = ,9494$.

EX-

EXAMPLE III.

If $N=7{,}333$. and $r=1{,}04$:
Then ,$N=7{,}62632$; and $\left(\frac{7{,}62632}{8{,}333}=\right)\frac{a}{a+b}=,9151$.

QUESTION LXIV.

Suppoſing the decrements of life to be equal, the preſent value of an annuity of 1 *l.* to continue during the joint lives of two perſons, whoſe reſpective complements of life are *m*, and *n* (*m* being the greater number) is required?

SOLUTION.

Since the probability of the continuance of the life, whoſe complement is n, for one year, is $\frac{n-1}{n}$, per qu. 56. and that of the life, whoſe comp. is m, $\frac{m-1}{m}$, per ditto; alſo, ſince theſe two events are independent on each other, it will follow (from queſt. 28) that $\frac{n-1}{n}\times\frac{m-1}{m}$ will be the probability of both of thoſe lives continuing in being one year; and by reaſoning in the ſame way $\frac{n-2}{n}\times\frac{m-2}{m}$ will be the probability thereof the ſecond year; $\frac{n-3}{n}\times\frac{m-3}{m}$ the third, &c. Which probabilities, being taken as the values of the ſeveral payments, which will become due at the end of the firſt, ſecond, third,

&c. years; and their preſent worths being found, as before, it will appear, that the value of the annuity will be expreſſed by the following ſeries,

$$\frac{\overline{n-1}\times\overline{m-1}}{nmr}+\frac{\overline{n-2}\times\overline{m-2}}{nmr^{2}}+\frac{\overline{n-3}\times\overline{m-3}}{nmr^{3}},\ \&c.$$

Of which ſeries *n* terms, only, will be uſeful; becauſe the life, whoſe complement is *n*, is ſuppoſed to be neceſſarily extinct in *n* years.

Now if the numerators of the fractions, which conſtitute the above ſeries, be expanded, by multiplication, they will appear as below,

$$\overline{n-1}\times\overline{m-1}=(nm-m-n+1=)\ nm-\overline{m+n}\times 1+1,$$
$$\overline{n-2}\times\overline{m-2}=(nm-2m-2n+4=)\ nm-\overline{m+n}\times 2+4,$$
$$\overline{n-3}\times\overline{m-3}=(nm-3m-3n+9=)\ nm-\overline{m+n}\times 3+9;$$
$$\&c.\qquad\qquad \&c.\qquad\qquad \&c.$$

And therefore the above ſeries may be divided into three other ſeries, *viz.*

$$\frac{nm}{nm}\times\overline{\frac{1}{r}+\frac{1}{rr}+\frac{1}{r^{3}}\,(n)}=\frac{nm}{nm}\times\frac{1-p}{r-1}\ \text{by queſt. 15.}$$

(where *p* is the preſent worth of 1 *l.* due at the end of *n* years)

$$-\frac{m+n}{nm}\times\overline{\frac{1}{r}+\frac{2}{r^{2}}+\frac{3}{r^{3}}\,(n)}=-\frac{m+n}{nm}\times\overline{\frac{\overline{1-p}\times r}{\overline{r-1}^{2}}-\frac{np}{r-1}},$$

(by queſt. 16.

$$+\frac{1}{nm}\times\overline{\frac{1}{r}+\frac{4}{r^{2}}+\frac{9}{r^{3}}\,(n)}=\frac{1}{nm}\times\overline{\frac{\overline{1-p}\cdot\overline{r+1}\cdot r}{\overline{r-1}^{3}}-\frac{2nrp}{\overline{r-1}^{2}}-\frac{n^{2}p}{r-1}}.$$

(by queſt. 17.

The ſums of which three ſeries, being ranged according to their reſpective diviſors and ſigns, will ſtand as below,

$$\frac{nm}{nm}\times\frac{1-p}{r-1},$$

$$+\frac{m+n}{nm}\times\frac{np}{r-1}-\frac{m+n}{nm}\times\frac{\overline{1-p}\times r}{\overline{r-1}^{2}},$$

$$-\frac{1}{nm}\times\frac{nnp}{r-1}-\frac{1}{nm}\times\frac{2rnp}{\overline{r-1}^{2}}+\frac{1}{nm}\times\frac{\overline{1-p}\cdot\overline{r+1}\cdot r}{\overline{r-1}^{3}}.$$

But $\frac{m+n}{nm}\times\frac{np}{r-1}-\frac{1}{nm}\times\frac{nnp}{r-1}=\frac{mn}{nm}\times\frac{p}{r-1},$

And $\frac{nm}{nm}\times\frac{1-p}{r-1}+\frac{nm}{nm}\times\frac{p}{r-1}=\frac{mn}{mn}\times\frac{1}{r-1}=\frac{1}{r-1};$

Alſo $\frac{m+n}{nm}\times\frac{\overline{1-p}\times r}{\overline{r-1}^{2}}+\frac{1}{nm}\times\frac{2rnp}{\overline{r-1}^{2}}=\frac{\overline{m+n}\times r-\overline{m-n}\times rp}{nm\times\overline{r-1}^{2}}:$

Therefore the value of the annuity will be

$$\frac{1}{r-1}-\frac{\overline{m+n}\times r-\overline{m-n}\times rp}{nm\times\overline{r-1}^{2}}+\frac{\overline{1-p}\times\overline{r+1}\times r}{nm\times\overline{r-1}^{3}}.$$

And putting $\frac{1}{r-1}=P$ (the value of the perpetuity) it will be

$$P-\frac{\overline{\overline{m+n}-\overline{m-n}\times p}\times r}{nm\times\overline{r-1}^{2}}+\frac{\overline{1-p}\times\overline{r+1}\times Pr}{nm\times\overline{r-1}^{2}};\ \text{Or}$$

$$P-\overline{\overline{m+n}-\overline{m-n}\times p}\times\frac{rPP}{nm}+\overline{1-p}\times\overline{r+1}\times P\times\frac{rPP}{nm};$$

Or $P-\overline{\overline{m+n}-\overline{m-n}\times p-\overline{1-p}\times\overline{r+1}\times P}\times\frac{rPP}{nm}.$

Although this may be the readieſt expreſſion to compute by, yet, for ſome ſubſequent purpoſes, it may be better to expreſs this value otherwiſe by putting $\frac{1}{r-1}=P$; $\frac{r}{\overline{r-1}^{2}}=Q$ and $\frac{rr+r}{\overline{r-1}^{3}}=R$ as in cor. queſt. 19. And then the value $\frac{1}{r-1}-\frac{\overline{m+n}\times r-\overline{m-n}\times rp}{nm\times\overline{r-1}^{2}}+\frac{\overline{1-p}\times\overline{r+1}\times r}{nm\times\overline{r-1}^{3}}$; will become $P-\frac{\overline{m+n}-\overline{m-n}\times p}{nm}Q+\frac{1-p}{nm}R$,

Or $$P+\frac{-m-n+mp-np}{nm}Q+\frac{1-p}{nm}R,$$

Or $$P+Q\times\overline{-\frac{1}{n}-\frac{1}{m}+\frac{p}{n}-\frac{p}{m}}+R\times\overline{\frac{1}{nm}-\frac{p}{nm}};$$

And (by reſtoring $\frac{1}{r^n}$ for p.)

$$P+Q\times\overline{-\frac{1}{n}-\frac{1}{m}+\frac{1}{nr^n}-\frac{1}{mr^n}}+$$
$$(R\times\overline{\frac{1}{nm}-\frac{1}{nmr^n}}.$$

EXAMPLE I.

What is the value of an annuity of 1 *l.* to continue during the joint lives of two perſons, of the reſpective ages of 43 and 54, ſuppoſing the decrements of life to be equal, and that compound intereſt be allowed at 4 *per Cent.*

Here $n=(86-54=)32$; $m=(86-43=)43$; $p=0,285058$; $r=1,04$; and $P=\frac{1}{0,04}=25$.

$$\begin{array}{lll}
\text{Then } p=0,285058; & 1-p=0,714942; & m+n=75 \\
m-n=11 & r+1=2,04 & \\
 & & -3.135638 \\
3.135638; & 1,458482; & -36,462050 \\
 & P=25 & \\
PP=625. & & \text{Rem. } 35,402312 \\
 & 36,462050; & r=1,04 \\
mn=1376 & & \\
 & & 36,818404
\end{array}$$

And $36,818404\times\frac{625}{1376}=16,723475$

Laftly $25-16,723475=8,276525$, the preſent worth of the annuity.

EXAMPLE II.

What is the preſent value of an annuity of 1 *l.* to continue during the joint Lives of two perſons, the complements of whoſe lives are ſeverally 43 and 20, allowing 4 *l. per Cent.* compound intereſt ?

Here $m=43$; $n=20$; $p=0,456387$; $r=1,04$; $P=25$.

$$\begin{array}{lll}
\text{Then } p=0,456387; & 1-p=0,543613; & m+n=63 \\
m-n=23 & r+1=2,04 & \\
 & & -10,496901 \\
10\,496901; & 1,108970; & -27,724250 \\
 & P=25 & \\
PP=625 & & \text{Rems. } 24,778849 \\
 & 27,724250; & r=1,04 \\
mn=860 & & \\
 & & 25,770008
\end{array}$$

And $25,770003\times\frac{625}{860}=18,7282$

Laftly $25-18,7282=6\,2718$ the anſwer.

EXAMPLE III.

What is the preſent value of an annuity of 1 *l.* for the joint lives of two perſons, whoſe complements of life are 32 and 20, allowing compound intereſt at 4 *l. per Cent.*

Here $m=32$; $n=20$; $p=0,456387$; $r=1,04$; $P=25$.
Then $p=0,456387$; $1-p=0,543613$; $m+n=52,$

$m-n=12$ $\quad r+1 \quad 2,04$ $\quad - 5,476644$

5,476644; $\quad$ 1,108970; $\quad -27,724250$

$P= \quad 25$

Rems. 18,799106;

$PP=625$ $\quad$ 27,724250; $\quad r= \quad 1,04$

$mn=640$ $\quad$ 19,551070;

And $19,551073\times\frac{625}{640} = 19,092842$

Laſtly $25-19,092842=5,907158$
the value of the annuity required.

SCHOLIUM.

The value of the above annuity may be obtained, nearly, by a ſhorter proceſs, if the preſent worth of an annuity on that ſingle life, whoſe complement is n, be a known number, as follows:

$$\text{The ſeries } \frac{\overline{n-1}\cdot\overline{m-1}}{nmr}+\frac{\overline{n-2}\cdot\overline{m-2}}{nmr^2}+\frac{\overline{n-3}\cdot\overline{m-3}}{nmr^3}\ (n),$$

which expreſſes the value of the annuity, may be conſidered as compoſed by multiplying the correſponding terms of the two following ſeries, together.

$$\frac{m-1}{m}+\frac{m-2}{m}+\frac{m-3}{m}+\frac{m-4}{m}\ (n),$$

And $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}+\frac{n-4}{nr^4}\ (n);$

The firſt of which is an arithmetical progreſſion, whoſe greateſt term is $\frac{m-1}{m}$, common difference $\frac{1}{m}$, and number of terms n; Therefore (by queſtion 17, part 2. vol. I.) the ſum thereof is $\left(\frac{n\times\overline{m-1}}{m}-\frac{n\cdot\overline{n-1}}{2\cdot m}=\right)$ $\frac{2n\times\overline{m-1}-n\times\overline{n-1}}{2m}$,

Or $\left(\frac{2nm-2n-nn+n}{2m}=\frac{2mn-nn-n}{2m}=\right)n-\frac{n\cdot\overline{n+1}}{2m}.$

And, although the ſecond ſeries (being the value of that ſingle life whoſe complement is n) is not ſtrictly ſpeaking an arithmetical progreſſion, yet as its ſum is given $=N$; and as r, the ratio of the geometrical progreſſion of diviſors, is not much greater than unity, it may be conſidered as nearly ſuch; and the common difference may be found per queſt. 6. part 2. vol. I. for the greateſt term is $\frac{n-1}{nr}$; and the leaſt term is $\frac{n-n}{nr^n}=0$; whoſe difference is $\frac{n-1}{nr}$; which, being divided by $n-1$, quotes $\frac{1}{nr}$, for the common difference thereof.

Now, by queſt. 21. the ſum of n terms of ſuch a ſeries of products may be found from the above data,

$=N$

$$=\frac{N}{n}\times\overline{n-\frac{\overline{n+1}\cdot n}{2m}+\frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{3\cdot 2\cdot 2}}\times\frac{1}{m}\times\frac{1}{mr};$$

Or $N\times\overline{1-\frac{n+1}{2m}+\frac{\overline{n+1}\cdot\overline{n-1}}{12mr}}$:

Where, if we put $\frac{n+1}{2m}=q$; it will be

$N-Nq+\frac{n-1}{6r}q$ or $N-\overline{Nq-\frac{n-1}{6r}q}$:

That is $N-\overline{N-\frac{n-1}{6r}}\times q$

E X A M P L E I.

What is the value of an annuity of 1 *l.* to continue during the joint lives of two perſons, of the reſpective ages of 43 and 54. *See* example I. *aforegoing*.

Here $m=43$; $n=32$; $r=1{,}04$; and $N=10{,}478$, by Example 2. queſt. 56.

Then $n-1=31$; $6r=6{,}24$; $n+1=33$; and $2m=86$;

6,24) 31,000 (4,9679, and 86) 33,000 (,38372 $=q$

$N=10{,}478$

$-\ 4{,}968$

$N=10{,}478$

And $5{,}510\times 0{,}38372=\ 2{,}114$

Remains 8,364 the anſwer.

Which anſwer differs from the true anſwer, before found, only by being 0,088 too much.

E X A M P L E II.

What is the value of two joint lives, whoſe complements are 43 and 20? *See* example II. *aforegoing*.

Here $m=43$; $n=20$; $r=1,04$; and $N=7,333$, by example 3. queſt. 56.

Now $n-1=19$; $6r=6,24$; $n+1=21$ and $2m=86$

6,24) 19,000 (3,045, and 86) 21,000 (,24418;

$N=7,333$

$-3,045$

$N=7,333$

And 4,288 × 0,24418=1,047

Remains 6,286 the anſwer.

Which is but 0,014 greater than the true anſwer.

EXAMPLE III.

What is the value of two joint lives, whoſe complements are 32 and 20? *See* example III. *aforegoing.*

Here $m=32$; $n=20$; $r=1,04$; $N=7,333$;

Now $n-1=19$; $6r=6,24$; $n+1=21$; and $2m=64$

6,24) 19,000 (3,045, and 64) 21,000 (,328125;

$N=7,333$

$-3,045$

$N=7,333$

4,288×0,328125=1,407

5,926 the anſwer.

Which is but 0,019 greater than the true anſwer; and none of the above differences are $\frac{1}{10}$ of a year's purchaſe.

Since the anſwers, obtained by this approximation, differ ſo little from the true anſwers; and ſince the proceſs is ſo much ſhorter; we may ſafely, in common caſes, make uſe thereof, for which reaſon, it is here annexed in words at length.

The Rule to find the preſent value of an annuity, which is to continue during the joint lives of two perſons of given ages; allowing compound intereſt at a given rate, ſuppoſing the decrements of life to be equal, and having the value of the oldeſt life given.

Let the number of years, which each of the perſons want of 86, *be called their complements of life; and let the ſum of* 1 *l. and its intereſt for* 1 *year, be called the rate.*

From the leſſer complement ſubtract one, and divide the remainder by ſix times the rate; or find this quotient, in table the laſt.

From the value of the life of the oldeſt perſon ſubtract the above found quotient, and multiply the remainder by the leſſer complement more one.

Divide the laſt found product by twice the greater complement, and ſubtract the quotient from the value of the oldeſt life, then the remainder will be the value of an annuity for the joint lives, which was required.

EXAMPLE.

What is the value of an annuity of 1 *l.* to continue during the joint lives of two perſons of the reſpective ages of 43 and 54; allowing compound intereſt at four *per Cent.* the value of the life of 54 years being 10,478?

Here (86—43=) 43, is the greater complement.

And (86—54=) 32, is the leſſer complement.

Alſo (1+,04=) 1,04 is the rate.

Now if (32— 1=) 31 be divided by (1,04×6=) 6,24, the quotient will be 4,968.

From 10,478, take 4,968, and there will remain 5,510.

Which multiplyed by (32+1=) 33 will produce 181,830.

If the laſt product 181,830, be divided by ($43\times2=$) 86, the quotient will be 2,114.

Which quotient, taken from 10,478, will leave 8,364, for the value of the annuity required.

QUESTION LXV.

What is the preſent value of an annuity of 1 *l.* to continue during the joint lives of two perſons, each aged 48 years, ſuppoſing the decrements of life to be equal, and that compound intereſt be allowed at 4 *per Cent.*

SOLUTION.

The ſolution of this queſtion may be deduced from that of queſt. 64; for if the two complements of life *m* and *n* be equal, it will follow that, $m+n=2n$; $\overline{m-n}\times p=0$; and $nm=nn$: Therefore the value of the annuity will be $\frac{1}{r-1}-\frac{2r}{n\times\overline{r-1}^{2}}$ $+\frac{\overline{1-p}\times\overline{r+1}\times r}{nn\times\overline{r-1}^{3}}$,

That is $P-\overline{2n-\overline{1-p}\times\overline{r+1}\times P}\times\frac{rPP}{nn}$:

Or in order to ſhew its connexion with ſome following ſolutions $\Big\}\, P-\frac{2}{n}Q+\frac{1-p}{nn}R.$

In this caſe,

$$\begin{array}{llllll}
n=(86-48=)\ 38; & r=1{,}04; & p=0{,}225285; & \text{and } P=25. \\
1-p= & {,}774715; & 2n=76, & & PP= & 625, \\
r+1= & 2{,}04 & & -39{\cdot}510475 & nn= & 1444; \\
\hline
& 1{,}580419; & \text{Rems.} & 36{,}489525, \\
P= & 25 & r= & 1{,}04 \\
\hline
& 39{\cdot}510475. & &
\end{array}$$

Now $37{,}949106 \times \frac{625}{1444} = 16{,}42534$

And $25 - 16{,}42534 = 8{,}57466$, the value required.

SCHOLIUM. But if N, the value of the ſingle life whoſe complement is n, be given; then, by the ſcholium to the laſt queſtion, the value of the annuity will be nearly,

$$N \times \overline{1 - \frac{n+1}{2n} + \frac{\overline{n+1} \times \overline{n-1}}{12nr}};\ \text{Or, becauſe } 1 - \frac{n+1}{2n} = \frac{n-1}{2n},$$

$$\left(N \times \frac{n-1}{2n} + \frac{\overline{n+1} \times n-1}{12nr} = \right)\ \overline{N + \frac{n+1}{6r}} \times \frac{n-1}{2n}:$$

Now by obſerving the operations of the laſt queſtion, it appears that the reſults, obtained by the above approximation, are greater than the truth: Therefore, when n is not a very ſmall number, we may ſafely write

$$\left(\frac{\overline{n+1} \times \overline{n-1}}{12r \times \overline{n+1}} = \right) \frac{n-1}{12r},\ \text{for } \frac{\overline{n+1} \times n-1}{12rn};\ \text{becauſe, there-}$$

by, the quotient will be diminiſhed a little; and then the annuity will be worth, nearly,

$$\left(N \times \frac{n-1}{2n} + \frac{n-1}{12r} = \right)\ \overline{\frac{N}{n} + \frac{1}{6r}} \times \frac{n-1}{2}.$$

Which, in words at length, follows:

The Rule for finding the preſent value of an annuity, to continue during the joint lives of two perſons of equal ages, having the value of the ſingle life given, ſuppoſing the decrements of life to be equal, and allowing compound intereſt at a given rate.

Let the number of years, which the given age wants of 86, be called the complement of life; and let the ſum of one pound, and its intereſt for one year, be called the rate. Divide the given value of the ſingle life, by the complement of life; alſo divide unity by ſix times the rate, (which laſt quotient is in the firſt line of table the laſt.)

Add thoſe two quotients together, and multiply their ſum by the complement leſs one; *then ſhall half the product be the value of the annuity required.*

EXAMPLE I.

If the given age of the two perſons be 48 years, and the rate of intereſt four *per Cent.* then the value of the ſingle life will be 11,748.

Then (86—48=) 38 will be the complement of life,

And (1+,04+) 1,04 will be the rate.

Then, if 11,748 be divided by 38, the quotient will be 0,30917, and if one be divided by (6×1,04=) 6,24, the quotient will be 0,16026.

Alſo if 0,30917 be added to 0,16026, and the ſum 0,46943 be multiplyed by (38—1=) 37 the product will be 17,3630; the half of which, *viz.* 8,6815, is the value of the annuity required.

EXAMPLE II.

If the two perſons be each 54 years of age, allowing compound intereſt at the ſame rate?

Then 10,478, will be the value of the ſingle life.

(86—54=) 32, the complement of life:

And (1+,04=) 1,04, the rate.

Now if 10,478, be divided by 32, the quotient will be 0,3275.

Then (0,3275+0,1603=) 0,4878 being multiplied by (32—1=) 31, prodnces 15,1218; the half of which, *viz.* 7,5609, is the value of the annuity required.

QUESTION LXVI.

Suppoſing the decrements of life to be in a conſtant ratio, let there be two lives, the probabilities of the continuance of which for one year are reſpectively x and y; It is required to find the preſent value of an annuity on their joint continuance, allowing compound intereſt to the purchaſer?

SOLUTION.

By the principles above given, the probability of both the lives continuing the firſt year will be xy, the ſecond year $xxyy$, the third year x^3y^3, &c.

Which expreſſions being the value of each payment; their preſent worths $\left(\frac{xy}{r}+\frac{xxyy}{rr}+\frac{x^3y^3}{r^3}, \&c.\right)$ will be the value of the annuity required.

Which is a geometrical progreſſion infinitely decreaſing (for x and y are both leſs than unity) whoſe greateſt term is $\frac{xy}{r}$; ratio $\frac{r}{xy}$; and ratio leſs unity $\left(\frac{r}{xy}-1=\right)$

$r-xy$

$\frac{r-xy}{xy}$; Therefore the ſum thereof will be $\frac{xy}{r-xy}$; which is the value of the annuity required.

QUESTION LXVII.

Having given the values of the annulties of two ſingle lives (computed at a known rate, *per Cent.*) to find the value of an annuity for their joint lives, allowing compound intereſt at the ſame rate, and ſuppoſing the decrements of life to be in a conſtant ratio?

SOLUTION.

Let the given values of the two annuities be *N* and *M*; and *r* the amount of 1 *l.* in one year.

Then, (ſince the decrements of life are in a conſtant ratio) the probabilities of the continuance of thoſe lives each for one year will be reſpectively $\frac{rN}{1+N}$ and $\frac{rM}{1+M}$ by queſtion 63, which probabilities being ſubſtituted, in the places of *x* and *y*, in the reſult of queſtion 66, (*viz.* $\frac{xy}{r-xy}$) it will then become, the quotient of $\frac{rN}{1+N} \times \frac{rM}{1+M}$ divided by $\left(r - \frac{rN}{1+N} \times \frac{rM}{1+M} =\right)$ $\frac{\overline{1+N} \times \overline{1+M} \times r - rN \times rM}{\overline{1+N} \times \overline{1+M}}$;

That is $\frac{rN}{1+N} \times \frac{rM}{1+M} \times \frac{\overline{1+N} \times \overline{1+M}}{\overline{1+N} \times \overline{1+M} \times r - rN \times rM}$; will be the value of the annuity required.

Which reduced becomes $\dfrac{rNM}{\overline{1+N}\times\overline{1+M}-rNM}$;

Which is the theorem given by the celebrated Mr. *De Moivre*, in the firſt edition of his treatiſe of annuities on lives; who argues thus:

"Although the decrements of life be not really in a "conſtant ratio; yet as the values of the two ſingle "lives are determined, the combinations of two or more "ſuch lives will be, nearly, the ſame with the combina- "tions of two or more lives whoſe decrements are in a "conſtant ratio."

EXAMPLES.

I. What is the value of an annuity, on two joint lives, which ſingly computed (at 4 *l. per Cent. per Annum*) are reſpectively worth 12,683 and 10,478?

Here 11,683 × 11,478 $=157{,}053$
And 12,683 × 10,478 × 1,04 $=138{,}208$

18,845)138,208(7,334

II. What is the value of an annuity on two joint lives, which ſingly computed (at 4 *l. per Cent. per Annum*) are reſpectively worth 12,683 and 7,333?

Here 13,683 × 8,333 $=114{,}020$
And 12,683 × 7.333 × 1,04 $=96{,}625$

17,395) 96,625 (5,554

III. What is the value of an annuity on two joint lives, which ſingly computed (at 4 *l. per Cent. per Ann.*) are reſpectively worth 10,478 and 7,333?

Here 11,478 × 8,333 = 95,746
And 10,478 × 7,333 × 1,04 = 79,909

15,837) 77,909 (5,045

SCHOLIUM.

If theſe reſults be compared with the anſwers, upon the principle of equal decrements of life (found by queſt. 64) the values of the ſingle lives, given in the above examples, being the ſame with the values of the ſingle lives of the ages therein mentioned, it will appear that this rule will give the values of joint lives conſiderably leſs than that.

For the value of 2 lives of the ages 43 and 54 is	8,276;
The ſame by the above - - - -	7,334;
Difference - - - - - - - - - - -	0,942.
The value of 2 lives of the ages 43 and 66 is	6,272;
The ſame by the above - - - - -	5,554;
Difference - - - - - - - - - - -	0,718.
The value of 2 lives of the ages 54 and 66 is	5,907;
The ſame by the above - - - - - -	5,045;
Difference - - - - - - - - - - -	0,862.

Which differences at a medium are about ⅘ of a year's purchaſe.

COROL,

COROL.

If the lives are of equal ages, then the value of the annuity on their joint continuance will be

$$\frac{rNN}{\overline{1+N}^2-rNN}$$

QUESTION LXVIII.

The preſent value of annuity of 1 *l.* to continue during the joint lives of two perſons, of the reſpective ages of 10 and 31 (at 4 *per Cent.*) according to the table of obſervations, deduced from the bills of mortality, of *London*, is required?

SOLUTION.

If the probabilities of the continuance of the reſpective two lives, for one, two, three, *&c.* years, be taken from the ſaid tables of obſervations, they will be as follow, *viz.*

For the life of 10 years $\frac{517}{524}$, $\frac{510}{524}$, $\frac{504}{524}$, *&c.*

For the life of 31 years $\frac{367}{376}$, $\frac{358}{376}$, $\frac{349}{376}$, *&c.*

And conſequently the probability of their joint continuance for one, two, three, *&c.* years, will be $\frac{517\times367}{524\times376}$, $\frac{510\times358}{524\times376}$, $\frac{504\times349}{524\times376}$, the preſent worths of which, *viz.*

$$\frac{517\times367}{524\times376\times1{,}04}+\frac{510\times538}{524\times376\times\overline{1{,}04}^{2}}+\frac{504\times349}{524\times376\times\overline{1{,}04}^{3}}$$

(63) will be the value of the annuity required.

However tedious the numerical operation, (deduced from the above) may appear, no demonſtrated method that will ſhorten it has been publiſhed: for which reaſon, it has been thought proper to compare the reſult thereof with the two approximations above given.

As this ſeries conſiſts of fractions, having two factors in the numerator, and three in the denominator (whereof two are invariable), the value of each fraction will be eaſily obtained by a logarithmic proceſs; previous to which, it will be neceſſary to find the logarithm of the product of the two invariable factors: thus,

Log. of 524=2,7193
Log. of 376=2,5752

Log. of 524×376=5,2945

No. taken from Tab. Obſerv.	Logarithms of thoſe Numbers.	No. taken from Tab. Obſerv.	Logarithms of thoſe Numbers.	Sums of thoſe Log. being the Log. of the Numerators.	Logarithms of the powers of (r=) 1,04.	The laſt added to 5,2945; being the Log. of the Denominators.	The diff. of the Log. of the Nu. and Den. being the Log. of the Fractions.	The No. to the laſt Log. being the values of the Fractions.
517	2,7135	367	2,5647	5, 2782	0,0170	5, 3115	1, 9667	0, 9259
510	2,7076	358	2,5539	5, 2615	0,0341	5, 3286	1, 9329	0, 8568
504	2,7024	349	2,5428	5, 2452	0,0511	5, 3456	1, 8996	0, 7936
498	2,6972	340	2,5315	5, 2287	0,0681	5, 3627	1, 8661	0, 7347
492	2,6920	331	2,5198	5, 2118	0,0852	5, 3797	1, 8321	0, 6794
486	2,6866	322	2,5079	5, 1945	0, 1022	5, 3967	1, 7978	0, 6278
480	2,6812	313	2,4955	5, 1767	0, 1192	5, 4138	1, 7630	0, 5794
474	2,6758	304	2,4829	5, 1587	0, 1363	5, 4308	1, 7279	0, 5344
468	2,6702	294	2,4683	5, 1385	0, 1533	5, 4478	1, 6907	0, 4906
462	2,6646	284	2,4533	5, 1179	0, 1703	5, 4649	1, 6531	0, 4499
455	2,6580	274	2,4378	5, 0958	0, 1874	5, 4819	1, 6139	0, 4110
448	2,6513	264	2,4216	5, 0729	0, 2044	5, 4989	1, 5740	0, 3749
441	2,6444	255	2,4065	5, 0509	0, 2214	5, 5160	1, 5350	0, 3428
434	2,6374	246	2,3909	5, 0283	0, 2385	5, 5330	1, 4954	0, 3129
426	2,6294	237	2,3747	5, 0041	0, 2555	5, 5500	1, 4541	0, 2845
418	2,6212	228	2,3579	4, 9791	0, 2725	5, 5671	1, 4121	0, 2583

410	2,6128	220	2,3424	4,9552	0,2896	5,5841	1,3711	0,2350
402	2,6042	212	2,3263	4,9305	0,3066	5,6011	1,3295	0,2135
394	2,5955	204	2,3096	4,9051	0,3236	5,6182	1,2870	0,1936
385	2,5855	196	2,2923	4,8778	0,3407	5,6352	1,2426	0,1748
376	2,5752	188	2,2742	4,8494	0,3577	5,6522	1,1972	0,1575
367	2,5647	180	2,2553	4,8200	0,3747	5,6693	1,1508	0,1415
358	2,5539	172	2,2355	4,7894	0,3918	5,6863	1,1032	0,1268
349	2,5428	165	2,2175	4,7603	0,4888	5,7033	1,0570	0,1140
340	2,5315	158	2,1987	4,7302	0,4258	5,7204	2,0099	0,1023
331	2,5198	151	2,1790	4,6988	0,4429	5,7374	2,9615	0,0915
322	2,5079	144	2,1584	4,6663	0,4599	5,7544	2,9119	0,0816
313	2,4955	137	2,1367	4,6322	0,4769	5,7715	2,8608	0,0726
304	2,4829	130	2,1139	4,5968	0,4940	5,7885	2,8083	0,0643
294	2,4683	123	2,0899	4,5582	0,5110	5,8055	2,7527	0,0566
284	2,4533	117	2,0682	4,5215	0,5280	5,8226	2,6990	0,0500
274	2,4378	111	2,0453	4,4831	0,5450	5,8396	2,6435	0,0440
264	2,4216	105	2,0212	4,4428	0,5621	5,8566	2,5862	0,0386
255	2,4065	99	1,9956	4,4021	0,5791	5,8787	2,5285	0,0338
246	2,3909	93	1,9685	4,3594	0,5962	5,8906	2,4687	0,0294
237	2,3747	87	1,9395	4,3142	0,6132	5,9077	2,4065	0,0255

Sum of 36 terms of the ſeries carried forwards—10,7038

No. taken from Tab. Obſerv.	Logarithms of theſe Numbers.	No. taken from Tab. Obſerv.	Logarithms of thoſe Numbers.	Sums of thoſe Log. being the Log. of the Numerators.	Logarithms of the powers of (r=) 1,04.	The laſt added to5,2945; being the Log. of the Denominators.	The diff. of the Log of the Nu. and Den. being the Log. of the fractions.	The No. to the laſt Log. being the values of the fractions.
228	2,3579	81	1,9085	4,2664	0,6302	5,9248	2,3417	0,0220
220	2,3424	75	1,8751	4,2175	0,6473	5,9418	2,2757	0,0189
212	2,3263	69	1,8388	4,1651	0,6643	5,9588	2,2063	0,0161
204	2,3096	64	1,8062	4,1158	0,6813	5,9759	2,1400	0,0138
196	2,2923	59	1,7709	4,0632	0,6984	5,9929	2,0703	0,0118
188	2,2742	54	1,7324	4,0066	0,7154	6,0099	3,9967	0,0099
180	2,2553	49	1,6902	3,9455	0,7324	6,0270	3,9186	0,0083
172	2,2355	45	1,6532	3,8887	0,7495	6,0440	3,8447	0,0070
165	2,2175	41	1,6128	3,8303	0,7665	6,0610	3,7693	0,0059
158	2,1987	38	1,5798	3,7785	0,7835	6,0781	3,7005	0,0050
151	2,1790	35	1,5441	3,7231	0,8006	6,0951	3,6281	0,0042
144	2,1584	32	1,5052	3,6636	0,8176	6,1121	3,5515	0,0036
137	2,1367	29	1,4624	3,5991	0,8346	6,1292	3,4700	0,0030
130	2,1139	26	1,4150	3,5289	0,8516	6,1462	3,3827	0,0024
123	2,0899	23	1,3617	3,4516	0,8687	6,1632	3,2884	0,0019
117	2,0682	20	1,3010	3,3692	0,8857	6,1803	3,1890	0,0015

111	2,0453	17	1,2304	3,2757	0,9028	6,1973	3,0785	0,0012
105	2,0212	14	1,1461	3,1673	0,9198	6,2143	4,9510	0,0009
99	1,9956	12	1,0792	3,0748	0,9368	6,2314	4,8435	0,0007
93	1,9685	10	1,0000	2,9685	0,9539	6,2484	4,7201	0,0005
87	1,9395	8	0,9031	2,8426	0,9709	6,2654	4,5772	0,0004
81	1,9085	6	0,7782	2,6867	0,9879	6,2825	4,4043	0,0003
75	1,8751	5	0,6990	2,5741	1,0050	6,2995	4,2746	0,0002
69	1,8388	4	0,6021	2,4409	1,0220	6,3165	4,1245	0,0001
64	1,8062	3	0,4771	2,2833	1,0390	6,3336	5,9498	0,0001
59	1,7799	2	0,3010	2,0719	1,0561	6,3506	5,7213	0,0001
54	1,7324	1	0,0000	1,7324	1,0731	6,3676	5,3648	0,0000

Sum of the laſt 27 terms of the ſeries — — — 0,1398

Sum of the firſt 36 ditto ——— ——— 10,7038

Value of the annuity — 10,8436

Note, The Indexes of the laſt column of logarithms are negative, and were mark'd as ſuch in the copy; but the author, being at a great diſtance from the preſs, hopes this advertiſement to the reader will excuſe the printer's omiſſion, who muſt otherwiſe have alter'd the meaſure of his page, by which errors of greater conſequence might have happened, or the publication have been retarded.

The

Mr. *Simpſon* has obliged the world with tables of the values of lives calculated from the *London* obſervations, in which the value of the ſingle lives of the ages of 10 and 31 (at 4 *per Cent.*) are ſeverally 16,4, and 12,9; which being uſed for the ſymbols M and N in queſt. 67. will give Mr. *De Moivre*'s approximation to the values of the above joint lives, *viz.*

$$17{,}4\times13{,}9 = 241{,}86$$
$$16{,}4\times12{,}9\times1{,}04 = 220{,}02$$

21,84) 220,02 (10,07 the value of the annuity; which, if compared with the above, is above $\frac{3}{4}$ of a year's purchaſe too little.

Again, if 12,9 be uſed for N, in the rule given in the ſcholium to queſtion 64; that approximation to the value of the joint lives may ſtand thus.
$(86-10=)76=m$; $(86-31=)55=n$; And $(1,+0{,}04=)$ $1{,}04=r$ $1{,}04\times6=6{,}24$; And 6,24) 54,00 (8,6; then
$12{,}9-8{,}6=4{,}3$;

Alſo $(55+1=)$ $56\times4{,}3=240{,}8$; and $(2\times76=)$ 152: Then 152) 240,8 (1,6.

Laſtly $12{,}9-1{,}6=11{,}3$, the value of the annuity; which is $\frac{1}{2}$ a year's purchaſe too much.

SCHOLIUM.

It will be very eaſy to account for the former approximation being nearer to, and the latter farther from, the truth, in this caſe, than when the decrements of life were ſuppoſed equal; for the former, which is founded on the number of the living being in a geometrical progreſſion, is now better adapted to the numbers in the tables of obſervations (which the farther they decline from being truly arithmetical, the nearer they will neceſſarily approach to the being geometrical) than it was when the number of the living were ſuppoſed to be an arithmetical progreſſion: And on the contrary, the latter approximation (being founded on the ſummation of the ſeries of products, made by the ſeparate terms of two arithmetical progreſſions) muſt neceſſarily be farther from the truth, when neither of thoſe progreſſions is ſtrictly arithmetical, than when but one of them was otherwiſe.

However, when we conſider, in the queſtion before us, that the life of 31 as truly found by the table of obſervations, and that the numbers uſed in the life of 10 (which may be ſuppoſed to be as young as will commonly occur in practice) contain, greater deviations from an arithmetical progreſſion, than any elder age can; we may conclude, that (when the value of the younger life is computed from the tables) this approximation will never exceed the truth, by more than ½ a year's purchaſe.

And if we farther conſider, that the bills of mortality of *London* make the value of a ſingle life leſſer than the other methods of computation do; we may conclude, that ſuch exceſs will not exceed the real value of the joint lives of ſuch perſons as live out of that metropolis; or even in it, if they live temperately.

Mr. *Simpſon*, has given a very eaſy approximation (by the help of the tables of equal joint lives, inſerted in his treatiſe) which in the ſolution of this example comes extremely near the truth; but as he has thought proper to conceal the means he uſed to attain it, and as it would take up a great deal of time and labour to try it in other inſtances, I am, much againſt my will, obliged to omit it, together with many others (therein contained) which if true, are equally curious and uſeful: And the very ſame thing happens, with regard to the rule given for this purpoſe, by Mr. *De Moivre*, in the ſecond and third editions of his treatiſe of annuities on lives; which, he therein ſays, was originally derived from that above given in queſt. 67: As I have the honour to be intimately acquainted with him, I deſired that he would be ſo good to give me the inveſtigation thereof, in order to have inſerted it; which he readily promiſed to do; but has ſince told me, that he cannot find it among his papers, and that he doth not retain the manner of doing it, in his memory.

QUESTION LXIX.

Suppoſing the decrements of life to be equal, the preſent value of an annuity of 1 *l.* to continue during the joint lives of three perſons, whoſe reſpective complements of life are t, m, and n (t being $\sqsubset$ m and m $\sqsubset$ n) is required?

SOLUTION.

Since the probability of the continuance of the life, whoſe complement is n, for one year is $\frac{n-1}{n}$ *per* queſt. 56,

That of the life, whoſe complement is m, $\frac{m-1}{m}$ *per* ditto,

That of the life, whoſe complement is t, $\frac{t-1}{t}$ *per* ditto;

And, ſince theſe three events are independent on each other, it will follow, that $\frac{n-1}{n} \times \frac{m-1}{m} \times \frac{t-1}{t}$ will be the probability of all the three lives continuing one year; and by a like manner of reaſoning $\frac{n-2}{n} \times \frac{m-2}{m} \times \frac{t-2}{t}$ will be the probability of their continuing the ſecond year; $\frac{n-3}{n} \times \frac{m-3}{m} \times \frac{t-3}{t}$ the third year, *&c.*

Which probabilities, being taken as the values of the ſeveral payments, which will become due at the end of the firſt, ſecond, third, *&c.* years; and their preſent worths being found, as in queſt. 56, it will appear that the value of the annuity may be expreſſed by

$$\frac{\overline{n-1} \cdot \overline{m-1} \cdot \overline{t-1}}{nmtr} + \frac{\overline{n-2} \cdot \overline{m-2} \cdot \overline{t-2}}{nmtr^2} + \frac{\overline{n-3} \cdot \overline{m-3} \cdot \overline{t-3}}{nmtr^3}, \text{ &c.}$$

Of which ſeries, n terms, only, will be uſeful; becauſe the life, whoſe complement is n, is ſuppoſed to be neceſſarily extinct in n years.

Now, if the numerators of the terms of the above ſeries be expanded, by the actual multiplication of their ſeveral factors, they will become, *viz.*

$$\overline{n-1} \cdot \overline{m-1} \cdot \overline{t-1} = nmt - \overline{nm+nt+mt} \times 1 + \overline{n+m+t} \times 1 - 1,$$
$$\overline{n-2} \cdot \overline{m-2} \cdot \overline{t-2} = nmt - \overline{nm+nt+mt} \times 2 + \overline{n+m+t} \times 4 - 8,$$
$$\overline{n-3} \cdot \overline{m-3} \cdot \overline{t-3} = nmt - \overline{nm+nt+mt} \times 3 + \overline{n+m+t} \times 9 - 27,$$
&c. *&c.*

And therefore the above ſeries may be divided into 4 other ſeries, *viz.*

$$\frac{nmt}{nmt}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}\,(n)},$$

$$-\frac{nm+nt+mt}{nmt}\times\overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}\,(n)},$$

$$+\frac{n+m+t}{nmt}\times\overline{\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}\,(n)},$$

And $-\frac{1}{nmt}\times\overline{\frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}+\frac{64}{r^4}\,(n)};$

Which ſeries, being ſumm'd by queſtion 15, 16, 17, and 18, ſeverally become

$\frac{nmt}{nmt}\times\frac{1-p}{r-1},$ (where p is the preſent worth of 1 *l.* due at the end of n years.)

$$-\frac{nm+nt+mt}{nmt}\times\overline{\frac{\overline{1-p}\times r}{\overline{r-1}^{2}}-\frac{np}{r-1}},$$

$$+\frac{n+m+t}{nmt}\times\overline{\frac{\overline{1-p}\times\overline{r+1}\times r}{\overline{r-1}^{3}}-\frac{2nrp}{\overline{r-1}^{2}}-\frac{nnp}{r-1}},$$

$$-\frac{1}{nmt}\times\overline{\frac{\overline{1-p}\times\overline{\overline{r+2}^{2}\cdot r-3r}}{\overline{r-1}^{4}}-\frac{\overline{r+1}\times 3nrp}{\overline{r-1}^{3}}-\frac{3n^2rp}{\overline{r-1}^{2}}-\frac{n^3p}{r-1}};$$

Which ſums, being ranged according to their reſpective diviſors and ſigns may ſtand as follows.

$$\frac{nmt}{nmt}\times\frac{1-p}{r-1},$$

$$+\frac{nm+nt+mt}{nmt}\times\frac{np}{r-1}-\frac{nm+nt+mt}{nmt}\times\frac{\overline{1-p}\times r}{\overline{r-1}^{2}},$$

$$-\frac{n+m+t}{nmt}\times\frac{nnp}{r-1}-\frac{n+m+t}{nmt}\times\frac{2nrp}{\overline{r-1}^{2}}+$$

$$+\frac{1}{nmt}\times\frac{n^3p}{r-1}+\frac{1}{nmt}\times\frac{3n^2rp}{\overline{r-1}^{2}}+$$

$$\Bigg(\frac{n+m+t}{nmt}\times\frac{\overline{1-p}\times\overline{r+1}\times r}{\overline{r-1}^{3}}$$

$$\Bigg(\frac{1}{nmt}\times\frac{\overline{r+1}\times 3rnp}{\overline{r-1}^{3}}-\frac{1}{nmt}\times\frac{\overline{\overline{1-p}\times\overline{r+2}^{2}\times r-3r}}{\overline{r-1}^{4}}.$$

To consider theſe in their order, let us begin with the numerators of thoſe four fractions, whoſe common denominator is $nmt\times\overline{r-1}$:

$$+nmt\times\overline{1-p}=nmt-nmtp,$$

$$+\overline{nm+nt+mt}\times np=nmtp+n^2mp+n^2tp,$$

$$-\overline{n+m+t}\times nnp=-n^2mp-n^2tp-n^3p,$$

And $+1\times n^3p=+n^3p;$

The ſum of which is nmt;

Th. $\left(\frac{nmt}{nmt}\times\frac{1}{r-1}=\right)\frac{1}{r-1}$ will be the value of theſe four fractions.

Let us next conſider theſe three fractions, whoſe common denominator is $nmt\times\overline{r-1}^{2}$;

Hère, if we put $nm+nt+mt=K$,

$$K-S\times\overline{r-rp}=-Kr+nmrp+ntrp+mtrp,$$

$$-\overline{n+m+t}\times 2rnp=\quad -2nmrp-2ntrp\quad -2rn^2p,$$

And $\quad +1\times 3rn^2p= \qquad\qquad +3rn^2p;$

The ſum of which is $=-Kr-nmrp-ntrp+mtrp+rn^2p$,

Or $-\overline{nm+nt+mt}\times r-\overline{m+t}\times nrp+\overline{mt+nn}\times rp$,

Or $-\overline{nm+nt+mt}\times r-\overline{m+t-n}\times nrp+mtrp$,

Or $-\overline{nm+nt+mt}\times r+\overline{mt-\overline{m+t-n}\times n}\times rp$;

Therefore the value of theſe three fractions will be,

$$-\frac{\overline{nm+nt+mt}\times r-\overline{mt-\overline{m+t-n}\times n}\times rp}{nmt\times\overline{r-1}^2},\ \text{Or}$$

$$-\overline{nm+nt+mt-\overline{mt-\overline{m+t-n}\times n}\times p}\times\frac{r}{nmt\times\overline{r-1}^2}.$$

But for the readier uſe of this hereafter, we may proceed thus; the ſum of the three fractions whoſe denominators are $nmt\times\overline{r-1}^2$ is

$$\frac{\overline{-nm+nt+mt}\times r-nmrp-ntrp+mtrp+rn^2p}{nmt\times\overline{r-1}^2},$$

Or $\dfrac{-nm-nt-mt-nmp-ntp+mtp+nnp}{nmt}\times\mathcal{Q}$ (by writing $\mathcal{Q}$ for $\dfrac{r}{\overline{r-1}^2}$):

That is $Q\times\overline{-\frac{1}{t}-\frac{1}{m}-\frac{1}{n}-\frac{p}{t}-\frac{p}{m}+\frac{p}{n}+\frac{np}{mt}}$,

Or $Q\times\overline{-\frac{1}{t}-\frac{1}{m}-\frac{1}{n}-\frac{1}{tr^n}-\frac{1}{mr^n}+\frac{1}{nr^n}+\frac{n}{mtr^n}}$, by restoring $\frac{1}{r^n}$ for p.

The two fractions, whoſe common denominator is $nmt\times\overline{r-1}^3$ may (putting $n+m+t=s$) be added together as follows,

$$+s\times\overline{1-p}\times\overline{r+1}\cdot r=\overline{r+1}\cdot rs-\overline{r+1}\cdot rp\times\overline{n+m+t},$$
$$+1\times 3np\times\overline{r+1}\cdot r=+\overline{r+1}\cdot rp\times 3n,$$

The ſum of which is $=\overline{\overline{r+1}\cdot rs-\overline{r+1}\cdot rp\times\overline{m+t-2n}}$;

Or $\overline{n+m+t-\overline{m+t-2n}\times p}\times\overline{r+1}\cdot r$;

Therefore the value of thoſe two fractions will be

$$+\frac{\overline{n+m+t-\overline{m+t-2n}\times p}\times\overline{r+1}\times r}{nmt\times\overline{r-1}^3}.$$

This may, alſo, be otherwiſe expreſſed, for the uſe of ſubſequent ſolutions, thus (putting $\frac{\overline{r+1}\times r}{\overline{r-1}^3}=R$)

$$\frac{n+m+t-\overline{m+t-2n}\times p}{nmt}\times R,$$

Or $R\times\frac{n+m+t-mp-tp+2np}{nmt}$;

That is $R\times\overline{\frac{1}{mt}+\frac{1}{nt}+\frac{1}{nm}-\frac{p}{nt}-\frac{p}{nm}+\frac{2p}{mt}}$;

Or $R\times\overline{\frac{1}{mt}+\frac{1}{nt}+\frac{1}{nm}-\frac{1}{ntr^n}-\frac{1}{nmr^n}+\frac{2}{mtr^n}}$, by reſtoring $\frac{1}{r^n}$ for p.

And the value of the annuity will be,

$$\frac{1}{r-1}-\overline{nm+nt+mt-\overline{mt-\overline{m+t-n}\times n}\times p}\times\frac{r}{nmt\times\overline{r-1}^2}$$

$$+\frac{\overline{\overline{n+m+t}-\overline{m+t-2n}\times p}\times\overline{r+1}\times r}{nmt\times\overline{r-1}^3}$$

$$-\frac{\overline{1-p}\times\overline{\overline{r+2}^2\times r-3r}}{nmt\times\overline{r-1}^4}.$$

The numerical operation deduced from this expreſſion, contracted as below, will be the moſt expeditious; but the following (conſiſting of the two before-found expreſſions, with the ſubſtitution of $\frac{\overline{r+2}^2\times r-3r}{\overline{r-1}^4}=S$) will hereafter be wanted, *viz.*

$$P+Q\times\overline{-\frac{1}{t}-\frac{1}{m}-\frac{1}{n}-\frac{1}{tr^n}-\frac{1}{mr^n}+\frac{1}{nr^n}+\frac{n}{mtr^n}}$$

$$+R\times\overline{\frac{1}{mt}+\frac{1}{nt}+\frac{1}{nm}-\frac{1}{ntr^n}-\frac{1}{nmr^n}+\frac{2}{mtr^n}}.$$

$$=S\times\frac{1-p}{nmt}.$$

Now, in order to ſhorten the firſt expreſſion, ſubſtitute $K = nm + nt + mt$, and $n + m + t = s$, as before.

Alſo $\frac{1}{r-1} = P$; $m + t = w$; and $\left(\frac{r}{nmt \times \overline{r-1}^{2}}\right)$

$\frac{PPr}{mnt} = L$: Then will, the annuity be equal to

$$P - \overline{K - \overline{mt - \overline{w-n}\times n}\times p}\times L$$
$$+ \overline{s - \overline{w - 2n}\times p}\times\overline{r+1}\times p\times L - \overline{1-p}\times\overline{\overline{r+2}^{2} - 3}\times PP\times L,$$

$$\text{Or } P - L\times\begin{cases} \overline{K - \overline{mt - \overline{w-n}\times n}\times p} \\ -\overline{s - \overline{w-2n}\times p}\times\overline{r+1}\times P \\ +\overline{1-p}\times\overline{\overline{r+2}^{2} - 3}\times PP; \end{cases}$$

$$\text{Or } P - L\times\begin{cases} K - \overline{mt - \overline{w-n}\times n}\times p \\ -P\times\begin{cases} \overline{s - \overline{w-2n}\times p}\times\overline{r+1} \\ -\overline{1-p}\times\overline{\overline{r+2}^{2} - 3}\times P. \end{cases} \end{cases}$$

EXAMPLE.

What is the preſent value of an annuity of 1 *l.* to continue during the joint lives of three perſons, whoſe ages are 43, 54, and 66, allowing compound intereſt at 4 *l. per Cent*.

Here $n=(86-66=)\ 20$; $m=(86-54=)\ 32$; and $t=(86-43=)\ 43$.

And $r=1{,}04$; $P=25$; $w=(32+43=)\ 75$; $p=0{,}456387$.

$$\begin{array}{llllll}
n=20; & nm=640; & mt=1376; & r+1= & 2{,}04; & w=75. \\
m=32; & nt=860; & \overline{w-n}\times n=1100; & r+2= & 3{,}04; & -2n=40. \\ \cline{3-3} \cline{5-6}
t=43; & mt=1376; & & & & \\ \cline{1-2}
 & & \text{Rems.}\ 276; & \overline{r+2}^{2}= & 9{,}2416; & 35. \\ \cline{3-3}
s=95; & K=2876; & & = & 3{,}0000. & \\
 & & PP=625; & & & \\ \cline{5-5}
 & & & \text{Rems.} & 6{,}2416. &
\end{array}$$

Now $p=$ 0,456387;	$1-p=$ 0,543613;	$p=$ 0,456387.
276	6,2416	35.
125,962812.	3,393015.	15,973545.
	25	
$K=$2876		$s=$95
− 125,962812	84,825375.	−15,973545
Rems. 2750,037188	161,213968	Rems 79,026455
−1909,714825	− 84,825375	2,04
840,322363	76,388593.	161,213968.
	25	
$PP=$ 625		$m=$ 32
$r=$1,04	1909,714825	$t=$ 43
650		1376
		$n=$ 20
		27520.

$$\therefore \text{ And } 840{,}322363 \times \frac{650}{27520} = 19{,}84775.$$

Then (25—19,84775=) 5,15225, will be the preſent value of the annuity.

SCHOLIUM.

Since the above proceſs is very tedious, it may be an agreeable ſervice to ſhorten the ſame, by an approximation, ſimilar to that in the ſcholium to queſt. 64.

The ſeries above given for the value of this annuity, viz. $\frac{\overline{n-1}\cdot\overline{m-1}\cdot\overline{t-1}}{nmtr}+\frac{\overline{n-2}\cdot\overline{m-2}\cdot\overline{t-2}}{nmtr^2}+\frac{\overline{n-3}\cdot\overline{m-3}\cdot\overline{t-3}}{nmtr^3}$ (*n*), may be conſidered as compoſed of the products of the correſponding terms of the three following ſeries,

$\frac{t-1}{t}$

$$\frac{t-1}{t}+\frac{t-2}{t}+\frac{t-3}{t}+\frac{t-4}{t}\ (n),$$

$$\frac{m-1}{m}+\frac{m-2}{m}+\frac{m-3}{m}+\frac{m-4}{m}\ (n),$$

$$\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}+\frac{n-4}{nr^4}\ (n);$$

The two firſt of which are arithmetical progreſſions, and the laſt (whoſe ſum we ſuppoſe to be the known quantity N) may be taken as ſuch, without a conſiderable error.

In the

	ſeries, the greateſt term is	the common difference	and the ſum of the n terms
firſt	$\frac{t-1}{t}$	$\frac{1}{t}$	$n-\frac{\overline{n\cdot n+1}}{2t}$
ſecond	$\frac{m-1}{m}$	$\frac{1}{m}$	$n-\frac{\overline{n\cdot n+1}}{2m}$
third	$\frac{n-1}{nr}$	$\frac{1}{nr}$	$N.$

Hence the expreſſion, given in queſt. 34, for the ſum of n terms of ſuch a ſeries of products, *viz.*

$$\frac{SZW}{nn}+\frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}\times\overline{a\delta\Delta+bd\Delta+cd\delta}-\frac{\overline{n+1}\cdot n\cdot\overline{n-1}^{2}}{2\cdot2\cdot2}$$

$\times\, d\delta\Delta$, may be applied to this caſe, by making

$S=N$; $a=\frac{n-1}{nr}$; And $d=\frac{1}{nr}$:

$Z=n-\frac{n\cdot\overline{n+1}}{2m}$; $b=\frac{m-1}{m}$; And $\delta=\frac{1}{m}$:

$W=n-\frac{n\cdot\overline{n+1}}{2t}$; $c=\frac{t-1}{t}$; And $\Delta=\frac{1}{t}$.

Hence

$$\frac{SZW}{nn}=\left(S\times\frac{Z}{n}\times\frac{W}{n}=\right)N\times\overline{1-\frac{n+1}{2m}}\times\overline{1-\frac{n+1}{2t}},$$

$$a\delta\Delta=\frac{n-1}{nr}\times\frac{1}{m}\times\frac{1}{t}=\frac{n-1}{nmtr},$$

$$bd\Delta=\frac{m-1}{m}\times\frac{1}{nr}\times\frac{1}{t}=\frac{m-1}{nmtr},$$

$$cd\delta=\frac{t-1}{t}\times\frac{1}{nr}\times\frac{1}{m}=\frac{t-1}{nmtr},\text{ And}$$

$$d\delta\Delta=\frac{1}{nr}\times\frac{1}{m}\times\frac{1}{t}=\frac{1}{nmtr}.$$

$$\frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}\times\overline{a\delta\Delta+bd\Delta+cd\delta}\left\{\begin{array}{l}=\dfrac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}\times\\ \dfrac{\overline{n-1}+\overline{m-1}+\overline{t-1}}{nmtr}\\ =\dfrac{\overline{n+1}\cdot\overline{n-1}}{2\cdot6}\times\\ \dfrac{n+m+t-3}{mtr},\end{array}\right.$$

$$\text{And }\frac{\overline{n+1}\cdot n\cdot\overline{n-1}^{2}}{2\cdot2\cdot2}\times d\delta\Delta\left\{\begin{array}{l}=\dfrac{\overline{n+1}\cdot n\cdot\overline{n-1}^{2}}{2\cdot2\cdot2}\times\dfrac{1}{nmtr}\\ =\dfrac{\overline{n+1}\times\overline{n-1}^{2}}{2\cdot4}\times\dfrac{1}{mtr}:\end{array}\right.$$

$$\text{But }\frac{\overline{n+1}\cdot\overline{n-1}}{2\cdot6}\times\frac{n+m+t-3}{mtr}=\frac{n+1}{2m}\times\frac{n-1}{2r}\times\frac{n+m+t-3}{3t}$$

$$\text{And }\frac{\overline{n+1}\cdot\overline{n-1}^{2}}{2\cdot4}\times\frac{1}{mtr}=\frac{n+1}{2m}\times\frac{n-1}{2r}\times\frac{n-1}{2t},$$

$$\left.\begin{array}{l}\text{The difference of}\\ \text{which will be}\end{array}\right\}\frac{n+1}{2m}\times\frac{n-1}{2r}\times\overline{\frac{n+m+t-3}{3t}-\frac{n-1}{2t}}:$$

But

$$\text{But } \frac{n+m+t-3}{3t} - \frac{n-1}{2t} \left\{ \begin{array}{l} = \dfrac{2n+2m+2t-6-3n+3}{6t}, \\ = \dfrac{2m+2t-n-3}{6t}; \end{array} \right.$$

Therefore the value of the annuity will be

$$N \times \overline{1-\frac{n+1}{2m}} \times \overline{1-\frac{n+1}{2t}} + \frac{n+1}{2m} \times \frac{n-1}{2r} \times \frac{2m+2t-n-3}{6t}:$$

In which expreſſion, if $\frac{n+1}{2m} = q$; it will become

$$N \times \overline{1-q} \times \overline{1-\frac{n+1}{2t}} + \frac{n-1}{6r} q \times \frac{\overline{m+t} \times 2 - \overline{n+3}}{2t}.$$

The above, expreſſed in words at length, follows:

The Rule, for finding the preſent value of an annuity, to continue during the joint lives of three perſons of given ages, allowing compound intereſt at a given rate, ſuppoſing the decrements of life to be equal, and the value of the oldeſt life to be given:

Let the number of years, that each of the perſons want of 86, *be called their complements of life, and let the ſum of* 1 l. *and its intereſt for one year, be called the rate.*

To the leaſt complement add one, and divide the ſum by the double of the next greater complement, calling the quotient q; *let the ſame dividend be divided by twice the greateſt complement, calling that quotient* Q:

Subtract each of theſe quotients from unity; multiply the remainders together, and their product by the value of the oldeſt life, calling the product p.

Add the two greater complements together, and double their ſum, from which ſubtract the leaſt complement and three; multiply the remainder by the firſt found quotient q, *and*

and that product by the number which in table the laſt, ſtands on a line with the leaſt complement leſs unity, *and under the rate of intereſt.*

Divide the laſt found product by twice the greateſt complement, and let the laſt found quotient be added to the above product p, *ſo ſhall the ſum be the value of the annuity required.*

EXAMPLE.

What is the preſent value of an annuity of 1*l.* to continue during the joint lives of three perſons, whoſe ages are 43, 54, and 66; allowing compound intereſt at 4*l per Cent. per Annum*, the value of the oldeſt life being 7,333.

Here (86—68=) 20 the leaſt complement,
(86—54=) 32 the next greater complement,
(86—43=) 43 the greateſt complement,
And (1+,04=) 1,04 is the rate.

If (20+1=) 21 be divided by (2 × 32=) 64, the quotient, *q*, will be ,328125; And if the ſame dividend, 21, be divided by (43×2=) 86, the quotient, *Q*, will be ,244186.

The firſt quotient (*q*=) ,328125, being ſubtracted from unity, leaves ,671875, and the laſt quotient (*Q*=),244186, being ſubtracted from unity, leaves ,755814; which remainders being multiplied together, and their product being multiplied by 7,333, produces 3,7237 the product *p*.

The two greater complements (32, and 43) being added together make 75, the double of which ſum is 150; from which, ſubtracting (the leaſt complement more three =) 23 the remainder will be 127; which remainder being multiplied by the firſt found quotient ($q=$) ,328125, and that product by 3,0449 (the number ſtanding againſt (20—1=) 19, and under 4 *per Cent.* in table the laſt) will produce 126,8863; which being divided by ($2\times43=$) 86, will quote 1,4754; to which adding the before found product 3,7237, their ſum 5,1991, will be the value of the annuity required.

QUESTION LXX.

Suppoſing the decrements of life to be equal, required the preſent value of an annuity of 1 *l.* to continue during the joint lives of three perſons, two of which are of the age of 54, and the other 66; allowing compound intereſt, at 4 *l. per Cent.*

This queſtion may be anſwered from queſtion 69, by $t=m$; for then

$nm+nt+mt=(nm+nm+mm=)\ \overline{2n+m}\times m$;
$m+t=2m; mt=mm$; And $mt-\overline{m+t-n}\times n=mm-\overline{2m-n}\times n$;
That is $mt-\overline{m+t-n}\times n=(mm-2mn+nn=)\ \overline{m-n}^2$;
$nmt=nm^2$; $n+m+t=n+2m$; And $m+t-2n=\overline{m-n}\times 2$;

Therefore the value of the annuity will be

$$\frac{1}{r-1}-\overline{\overline{2n+m}\times m-\overline{m-n}^2}\times p\times\frac{r}{nm^2\times\overline{r-1}^2}$$

$$+\frac{\overline{n+2m-2p\times\overline{m-n}}\times\overline{r+1}\times r}{nm^2\times\overline{r-1}^3}-\frac{\overline{1-p}\times\overline{\overline{r+2}^2\times r-3r}}{nm^2\times\overline{r-1}^4};$$

Which expreſſion (putting $\frac{1}{r-1}=P$; $\overline{m-n}=d$;

And $\frac{r}{nm^2\times\overline{r-1}^2}=\frac{rPP}{nm^2}=L$) will become,

$$P-L\times\left\{\begin{array}{l}\overline{2n+m}\times m-ddp,\\ -P\times\left\{\begin{array}{l}\overline{n+2m-2pd}\times\overline{r+1}\\ -\overline{1-p}\times\overline{\overline{r+2}^2-3}\times P.\end{array}\right.\end{array}\right.$$

S C H O L I U M.

Or, if the approximation (given in the ſcholium) be applied hereto, Then

$\frac{n+1}{2t}=\frac{n+1}{2m}=q$; $\overline{m+t}\times 2-\overline{n+3}=4m-\overline{n+3}$; & $6t=6m$;

And the value of the annuity will be

$$N\times\overline{1-q}^2+\frac{n-1}{6r}q\times\frac{4m-\overline{n+3}}{2m}.$$

If the above queſtion be anſwered by the firſt method; then $n=20$; $m=32$; $r=1,04$; $p=0,456387$; & $P=25$. $m-n=12=d$; $\overline{m+2n}\times m=2304$; and $ddp=65,7197$ Alſo $\overline{2n+m}\times m-ddp=2238,\ 280272$. $n+2m=84$; $2pd=10,953288$; $84-10,953288=73,046712$; And $73,046712\times 2,04=149,015292$: $1-p=0,543613$; $r+2=3,04$ and $\overline{r+2}^2-3=6,2416$. Then $0,543613\times 6,2416\times 25=84,825275$.

Now $149,015292-84,825275=64,190017$; $64,190017\times 25=1604,\ 750425$; $rPP=650$; and $nm^2=20480$:

Then $\overline{2238,280272-1604,750425}\times\frac{650}{20480}=20,1071$:

Laſtly 25—20,1071=4,8929, will be the value required.

The latter method, expreſſed in words at length, may ſtand as follows:

The rule to find the preſent value of an annuity to continue during the joint lives of three perſons, two of which are of equal ages, and the third of a greater age than either of them; allowing compound intereſt at a given rate, ſuppoſing the decrements of life to be equal, and the value of the oldeſt life to be given.

Let the number of years, which each of the perſons want of 86, *be called their complements of life; and let the ſum of* 1 l. *and its intereſt for one year, be called the rate.*

To the leaſt complement add one, and divide the ſum by twice the greater complement, calling the quotient q.

Subtract that quotient q *from unity, multiply the remainder by itſelf, and that product by the given value of the oldeſt life, calling that laſt product* p.

From four times the greater complement ſubtract the leſſer complement, and the number 3; *multiply the remainder by the number, which in the laſt table correſponds to the leſſer complement* leſs one, *and rate. Alſo multiply that product by the above found quotient* q; *then divide this laſt product by twice the greater complement.*

To the laſt found quotient, add the before found product p, *and the ſum will be the value of the annuity required.*

The ſame queſtion anſwered by this rule.

Here (86—66=) 20 the leaſt complement.

(86—54=) 32 the greateſt complement.

And (1+,04=) 1,04 the rate; alſo 7,333= value of the oldeſt life.

If (20+1=) 21 be divided by (2×32=) 64, the quotient *q* will be 0,328125.

The above quotient being taken from unity leaves 0,671875; which remainder multiplied by itſelf produces 0,451415, and that product multiplied by 7,333 will give 3,3102, which call p.

If from $(4\times32=)$ 128 be taken $(20+3=)$ 23, the remainder will be 105; Then $105\times3{,}0449\times0{,}328125=104{,}9062$; which product being divided by $(2\times32=)$ 64 = will quote 1,6391.

Now $1{,}6391+3{,}3102=4{,}9493$, the value of the annuity required.

QUESTION LXXI.

Suppoſing the decrements of life to be equal; required the preſent value of an annuity of 1 *l.* to continue during the joint lives of three perſons, two of which are of the age of 66, and the other 43; allowing compound intereſt at 4 *l. per Cent.*

This queſtion may be alſo anſwered from queſt. 69, by making $m=n$; for then,

$nm+nt+mt=nn+nt+nt=\overline{n+2t}\times n$;

$mt=nt$; $m+t-n=t$; And $\overline{m+t-n}\times n=tn$;

Th. $mt-\overline{m+t-n}\times n=nt-nt=0$; $nmt=n^2t$;

$n+m+t=2n+t$; And $m+t-2n=t-n$:

Therefore the value of the annuity will be,

$$\frac{1}{r-1}-\overline{n+2t}\times n\times\frac{r}{n^2t\times r-1^{2}}$$

$$+\frac{\overline{2n+t-\overline{t-n}\times p\times\overline{r+1}\times r}}{n^2t\times\overline{r-1}^{3}}-\frac{\overline{1-p}\times\overline{\overline{r+2}^{2}\times t-3r}}{n^2t\times\overline{r-1}^{4}}$$

Which expreſſion (putting $\frac{1}{r-1}=P$

And $\frac{r}{nnt\times\overline{r-1}^{2}}=\frac{PPr}{nnt}=L$) will become,

$$P-L\times\left\{\begin{array}{l}\overline{n+2t}\times n\\ -P\times\left\{\begin{array}{l}\overline{2n+t-\overline{t-n}\times p}\times\overline{r+1}\\ -\overline{1-p}\times\overline{\overline{r+2}^{2}-3}\times P.\end{array}\right.\end{array}\right.$$

S C H O L I U M.

If we apply the method given in the ſcholium to queſt. 69 hereto; Then

$\frac{n+1}{2m}=\frac{n+1}{2n}$; And $\frac{\overline{m+1}\times 2-\overline{n+3}}{6t}=\frac{2t+n-3}{6t}$;

Therefore the value of the annuity will be

$$N\times\overline{1-\frac{n+1}{2n}}\times\overline{1-\frac{n+1}{2t}}+\frac{n+1}{2n}\times\frac{n-1}{2r}\times\frac{2t+n-3}{6t}:$$

Now $1-\frac{n+1}{2n}=\frac{n-1}{2n}$; And $1-\frac{n+1}{2t}=\frac{2t-n-1}{2t}$;

Th. $N\times\overline{1-\frac{n+1}{2n}}\times\overline{1-\frac{n+1}{2t}}=N\times\frac{n-1}{2n}\times\frac{2t-n-1}{2t}$,

That is $N\times\overline{2t-n-1}\times\frac{n-1}{4nt}$:

Alſo $\frac{n+1}{2n}\times\frac{n-1}{2r}\times\frac{2t+n-3}{6t}=\frac{n+1}{6r}\times\overline{2t+n-3}\times\frac{n-1}{4nt}$;

Therefore the value of the annuity will be

$$\overline{N\times\overline{2t-n-1}+\frac{n+1}{6r}\times\overline{2t+n-3}}\times\frac{n-1}{4nt}.$$

The above queſtion performed by the firſt method; where $n=20$; $t=43$; $r=1,04$; $p=0,456387$; $P=25$; $n+2t=106$; $106\times20=2120$; $2n+t=83$; $t-n=23$; $23\times,456387=10,496901$, which taken from 83 leaves $72,503099$; And $72,503099\times2,04=147,906322$:

$\overline{1-p}=0,543613$; $\overline{r+2}^2-3=6,2416$;

And $0,543613\times6,2416\times25=84,8254$;

Then $\overline{147,9063-84,8254}\times25=1577,0225$;

Alſo $\overline{2120-1577,0225}\times\frac{650}{17200}=20,5195$.

Then $(25-20,5195=)4,4805$, is the value of the annuity required.

The latter method (expreſſed in words at length) may ſtand as follows.

The Rule to find the preſent value of an annuity, to continue during the joint lives of three perſons, two of which are of equal ages, and the third of a leſſer age than either of them; allowing compound intereſt at a given rate, ſuppoſing the decrements of life to be equal; and the value of one of the elder lives to be given.

Let the number of years, which each of the perſons want of 86, *be called their complements of life, and let the ſum of* 1 *l. and its intereſt for one year, be called the rate.*

From twice the greater complement, take the leſſer complement and one; multiply the remainder by the given value of the elder life, calling the product p.

To twice the greater complement, add the leſſer, and from their ſum ſubtract the number 3; *multiply the remainder by the number which, in the laſt table, correſponds to the leſſer* complement more one, *and rate.*

To this product, add the above found product p; *multiply the ſum by the leſſer* complement leſs one, *and divide the* *that product by four times the product of the two complements, ſo ſhall the quotient be the value of the annuity required.*

Now if the ſame example be work'd by this laſt Rule,

Then 7,333 is the value of one of the elder lives;
Alſo (86—66=) 20 will be the leaſt complement,
(86—43=) 43 will be the greateſt ditto,
And (1+,04=) 1,04 will be the rate.

Now (43× 2=) 86; and (86—20—1=) 65; Then (7.333×65=) 476,65 =p.

Again (2×43=) 86; and 86+20=106; Alſo 106—3=103; Then if 103 be multiplied by 3,3654 (the tabular number) it produces 346,635.

If the above found product 476,645, be added to the ſaid product 346,635; the ſum will be 823,280; which multiplied by (20—1=) 19 produces 15642,320; and that product being divided by (4×43×20=) 3440 quotes 4,5471, the value of the annuity required.

QUESTION LXXII.

Suppoſing the decrements of life to be equal; required the preſent value of an annuity of 1 *l.* to continue during the joint lives of three perſons, each 66 years of age; allowing compound intereſt at 4 *l. per Cent.*

This queſtion likewiſe may be anſwered from queſtion 69, by making $n=m=t$; for then,

$nm+nt+mt=3nn$; $mt-\overline{m+t-n}\times n=nn-nn=0$;
$n+m+t=3n$; $m+t-2n=0$; and $nmt=n^3$:

Therefore the value of the annuity will be

$$\frac{1}{r-1}-3nn\times\frac{r}{n^3\times\overline{r-1}^2}+3n\times\frac{\overline{r+1}\cdot r}{n^3\times\overline{r-1}^3}-\frac{\overline{1-p}\times\overline{\overline{r+2}^2\times r-3r}}{n^3\times\overline{r-1}^4},$$

$$\text{Or }\frac{1}{r-1}-\frac{3r}{n\times\overline{r-1}^2}+\frac{3r\times\overline{r+1}}{nn\times\overline{r-1}^3}-\frac{\overline{1-p}\times\overline{\overline{r+2}^2\times r-3r}}{n^3\times\overline{r-1}^4}.$$

Which expreſſion (if $\frac{1}{r-1}=P$; $\frac{r}{\overline{r-1}^2}=\mathcal{Q}$, &c. as in corol. to queſtion 19. will become

$$P-\frac{3}{n}\mathcal{Q}+\frac{3}{nn}R-\frac{1-p}{nnn}S.$$

If the method given in the ſcholium, be applied hereto; then,

$$\frac{n+1}{2m}=\frac{n+1}{2t}=\frac{n+1}{2n}=q;\text{ And }\frac{\overline{m+t}\times2-\overline{n+3}}{6t}=\frac{n-1}{2n}.$$

And the value of the annuity will be,

$$\left(N\times\overline{1-q}^2+\frac{n-1}{2r}q\times\frac{n-1}{2n}=\right)N\times\overline{1-q}^2+\frac{\overline{n-1}^2\times q}{4rn}:$$

$$\text{But }\overline{1-q}=\left(1-\frac{n+1}{2n}=\frac{2n-n-1}{2n}\right)\frac{n-1}{2n};$$

Whence the value of the annuity will become,

$$N\times\frac{\overline{n-1}^2}{4nn}+\frac{\overline{n-1}^2}{4rn}\times\frac{n+1}{2n},\text{ or }N\times\frac{\overline{n-1}^2}{4nn}+\frac{\overline{n-1}^2}{4nn}\times\frac{n+1}{2r}.$$

$$\text{That is }\overline{N+\frac{n+1}{2r}}\times\frac{\overline{n-1}^2}{4nn}.$$

Now ſince $\overline{n-1}^2 = nn - 2n + 1$, And $n\times\overline{n-2} = nn - 2n$; Therefore $n\times\overline{n-2}$ may be wrote for $\overline{n-1}^2$ for the ſame reaſon as in queſt. 65. and then the value of the annuity will become

$$\overline{N+\frac{n+1}{2r}}\times\frac{n\times\overline{n-2}}{4nn}, \text{ Or } \overline{N+\frac{n+1}{2r}}\times\frac{n-2}{4n}.$$

The above queſtion anſwered by the firſt method,

Then $P=25$; $Q=650$; $R=33150$; $\overline{1-p}\times S = 1378412$; And $n=20$:

$\frac{650\times3}{20}=97{,}5$; $\frac{33150\times3}{20\times20}=248{,}625$; And

$\frac{1378412}{20\times20\times20}=172{,}3015$.

$$\begin{array}{rr} 25 & 97{,}5 \\ 248{,}625 & 172{,}3015 \\ \hline \end{array}$$

Then 273,625 — 269,8015 = 3,8235, the value of the annuity required.

The latter method, expreſſed in words at length, may ſtand as follows.

The Rule to find the preſent value of an annuity, to continue during the joint lives of three perſons of the ſame age; the value of a ſingle life of that age being given; allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, *be called the complement of life; and let the ſum of* 1 l. *and it's intereſt for one year, be called the rate.*

To the complement of life add one, and divide the ſum by twice

twice the rate; or take three times the number which, in table the laſt, correſponds to the complement more one, *and rate.*

To the above quotient, or product, add the value of the ſingle life, and multiply their ſum by the complement leſs two; *divide the product by four times the complement, and the quotient will be the value of the annuity required.*

The above queſtion anſwered by this rule.

Here (86—66=)20, will be the given complement of life,

7,333, the value of the ſingle life.

And (1+,04=) 1,04, the rate.

Then (20+1=) 21 being divided by (2×1,04=) 2,08, will quote (or 3,3654 the tabular number being multiplied by 3 will produce) 10,096.

And (10,096+7,333=) 17,429, being multiplied by (20—2=) 18, produces 313,72; which being divided by (4×20=) 80, will give 3,9215, for the value of the annuity required.

QUESTION LXXIII.

Suppoſing the decrements of life to be in a conſtant ratio; let there be three lives, the probabilities of the continuance of which for one year are reſpectively x, y, and z; it is required to find the preſent value of an annuity, on their joint continuance, allowing compound intereſt to the purchaſer.

SOLUTION.

By reaſoning, as in queſt. 66, the annuity may be expreſſed by the ſeries

$$\frac{xyz}{r} + \frac{x^2y^2z^2}{rr} + \frac{x^3y^3z^3}{r^3}, \text{ \&c.}$$

whoſe ſum is $\frac{xyz}{r-xyz}$; found in the ſame manner as in that queſtion.

QUESTION LXXIV.

Having given the values of the annuities of three ſingle lives (computed at a known rate of a intereſt) to find the value of an annuity, for their joint continuance, allowing compound intereſt to the purchaſer?

SOLUTION.

Let the given values of the three annuities be *N*, *M*, and *F*; and *r* the amount of 1 l. in one year:

Then, ſuppoſing the decrements of life to be in a conſtant ratio, the probabilities of the continuance of thoſe lives for one year will be reſpectively $\frac{rN}{1+N}$, $\frac{rM}{1+M}$ and $\frac{rF}{1+F}$ by queſtion 63, which expreſſions, being ſubſtituted in the ſtead of x, y, & z, in the laſt queſtion, will give

$$\frac{rN}{1+N}\times\frac{rM}{1+M}\times\frac{rF}{1+F}\times\frac{\overline{1+N}\times\overline{1+M}\times\overline{1+F}}{\overline{1+N}\times\overline{1+M}\times\overline{1+F}\times r-r^3NMF}$$

Or $\frac{rrNMF}{\overline{1+N}\times\overline{1+M}\times\overline{1+F}-rrNMF}$, for the value of the annuity required.

What is the value of an annuity on three joint lives; which ſingly, computed at 4 *l per Cent.* are reſpectively worth 12,683; 10,478; and 7,333.

$$13,683\times 11,478\times 8,333 = 1308,7266$$
$$12,683\times 10,478\times 7,333\times\overline{1,04}^2 = 1054,0198$$

254,7068)1054,0198(4,1382.

SCHOLIUM.

If this reſult be compared with the anſwer found per queſt. 69; it will appear that this rule will give the value of the joint lives, conſiderably too little.

For the value of the 3 lives of the ages 43, 54, and 66 is — —	5,1523;
By the above — —	4,1382,
Difference — —	1,0141.

COROL. I.

If the lives are of equal ages; then the value of the annuity, on their joint continuance, will be

$$\frac{rrN^3}{\overline{1+N}^3 - rrN^3}.$$

COROL. II.

Since the value of two equal joint lives is $\frac{rNN}{\overline{1+N}^2 - rNN}$; and the value of three equal joint lives $\frac{rrN^3}{\overline{1+N}^3 - rrN^3}$; it will follow, that the value of m equal joint lives will be $\frac{r^{\overline{m-1}}N^m}{\overline{1+N}^m - r^{\overline{m-1}}N^m}$.

It may be thought, that (as in queſt. 68. the value of two joint lives was computed, from the numbers given in the table of obſervations, deduced from the bills of mortality of *London*, ſo) we here ſhould compute the value of three joint lives upon the ſame principle; but as we then found, that value to be between the reſults given by the two kinds of approximation, which have now been conſidered; we may reaſonably expect, that the ſame will happen in this caſe; and conſequently, that by taking a mean between them, we ſhall not be far from the truth.

QUESTION LXXV.

Suppoſing the decrements of life to be equal, the value of an annuity, on four equal joint lives, is required?

SOLUTION.

Let n denote the complement of each of thoſe equal lives, and then it will appear, by reaſoning as in the former queſtions, that the annuity will be worth

$$\frac{\overline{n-1}^4}{n^4r}+\frac{\overline{n-2}^4}{n^4r^2}+\frac{\overline{n-3}^4}{n^4r^3}+\frac{\overline{n-4}^4}{n^4r^4}\ (n),\ \text{Or}$$

$$\frac{n^4-4n^3+6n^2-4n+1}{nnnnr}$$

$$+\frac{n^4-8n^3+24n^2-32n+16}{nnnnrr}$$

$$+\frac{n^4-12n^3+54n^2-108n+81}{nnnnrrr}\ (n);$$

Which may be divided into the five following ſeries:

$$\frac{n^4}{n^4}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}\ (n)}\ -$$

$$\frac{4n^3}{n^4}\times\overline{\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}\ (n)}\ +$$

$$\frac{6n^2}{n^4}\times\overline{\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}\ (n)}\ -$$

$$\frac{4n}{n^4}\times\overline{\frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}+\frac{64}{r^4}+\frac{125}{r^5}\ (n)}\ +$$

$$\frac{1}{n^4}\times\overline{\frac{1}{r}+\frac{16}{r^2}+\frac{81}{r^3}+\frac{256}{r^4}+\frac{625}{r^5}\ (n)}$$

Which ſeries being ſummed by queſtion 15, 16, 17, &c. will ſeverally become

$\frac{n^4}{n^4}\times\frac{1-p}{r-1}$ (where p is the preſent worth of 1*l.* due at the end of n years.)

$$-\frac{4n^3}{n^4}\times\overline{\frac{\overline{1-p}\times r}{\overline{r-1}^{2}}-\frac{np}{r-1}},$$

$$+\frac{6n^2}{n^4}\times\overline{\frac{\overline{1-p}\times\overline{rr+r}}{\overline{r-1}^{3}}-\frac{2npr}{\overline{r-1}^{2}}-\frac{nnp}{r-1}},$$

$$-\frac{4n}{n^4}\times\overline{\frac{\overline{1-p}\times\overline{r^3+4r^2+r}}{\overline{r-1}^{4}}-\frac{3np\times\overline{rr+r}}{\overline{r-1}^{3}}-\frac{3nnpr}{\overline{r-1}^{2}}-\frac{n^3p}{r-1}},$$

$$+\frac{1}{n^4}\times\left\{\begin{array}{l}\frac{\overline{1-p}\times\overline{r^4+11r^3+11r^2+r}}{\overline{r-1}^{5}}\ \frac{4np\times\overline{r^3+4r^2+r}}{\overline{r-1}^{4}}\\ -\frac{6nnp\times\overline{rr+r}}{\overline{r-1}^{3}}-\frac{4n^3pr}{\overline{r-1}^{2}}-\frac{n^4p}{r-1};\end{array}\right.$$

Which ſums being ranged according to their factors will become

$$\frac{n^4\times\overline{1-p}}{n^4\times\overline{r-1}},$$

$$+\frac{4n^4p}{n^4\times\overline{r-1}}-\frac{4n^3\times\overline{1-p}\times r}{n^4\times\overline{r-1}^2},$$

$$-\frac{6n^4p}{n^4\times\overline{r-1}}-\frac{12n^3p\times r}{n^4\times\overline{r-1}^2}+\frac{6n^2\times\overline{1-p}\times\overline{rr+r}}{n^4\times\overline{r-1}^3},$$

$$+\frac{4n^4p}{n^4\times\overline{r-1}}+\frac{12n^3p\times r}{n^4\times\overline{r-1}^2}+\frac{12n^2p\times\overline{rr+r}}{n^4\times\overline{r-1}^3}$$

$$-\frac{n^4p}{n^4\times\overline{r-1}}-\frac{4n^3p\times r}{n^4\times\overline{r-1}^2}-\frac{6n^2p\times\overline{rr+r}}{n^4\times\overline{r-1}^2}$$

$$\left(\frac{4n\times\overline{1-p}\times\overline{r^3+4r^2+r}}{n^4\times\overline{r-1}^4},\right.$$

$$\left(\frac{4np\times r^3+4r^2+r}{n^4\times r-1^4}\right. \qquad \frac{\overline{1-p}\times\overline{r^4+11r^3+11r^2+r}}{n^4\times\overline{r-1}^5}:$$

Now in thoſe five fractions, whoſe denominators are $n^4\times\overline{r-1}$, all the numerators are multiples of n^4; and conſequently that quantity will vaniſh, and the denominator will be only $\overline{r-1}$: Alſo the numerators $\overline{1-p}$ $+4p$, $-6p$, $+4p$, and $-p$, being collected by addition, become unity; therefore the ſum of theſe five fractions will be $\frac{1}{r-1}$.

In thoſe four fractions, whoſe denominators are $n^4\times\overline{r-1}^2$, all the numerators are multiples of n^3; which being cancelled in both numerators and denominators,

tors, the denominators will become $n\times\overline{r-1}^2$: And the numerators (omitting the common factor r) $-4, +4p, -12p, +12p$, and $-4p$, being added together become -4; therefore the ſum of theſe four fractions will be

$$-\frac{4r}{n\times r-1^2}$$

In the three fractions, whoſe denominators are $n^4\times\overline{r-1}^3$ (having cancelled nn in both numerators and denominators) the factors $6-6p, +12p$, and $-6p$, being added become $+6$, therefore the ſum of thoſe three fractions will be $+\frac{6\times\overline{rr+r}}{nn\times\overline{r-1}^3}$.

In like manner the ſum of the two fractions whoſe denominators are $n^4\times\overline{r-1}^4$ will be

$$-\frac{4\times\overline{r^3+4r^2+r}}{n^3\times\overline{r-1}^4}.$$

Therefore (putting $\frac{1}{r-1}=P$; $\frac{r}{\overline{r-1}^2}=Q$;

$$\frac{r^2+r}{\overline{r-1}^3}=R;\ \frac{r^3+4r^2+r}{\overline{r-1}^4}=S;\ \frac{r^4+11r^3+11r^2+r}{\overline{r-1}^5}=T)$$

The value of the annuity required will be

$$P-\frac{4}{n}Q+\frac{6}{nn}R-\frac{4}{n^3}S+\frac{1-p}{n^4}T.$$

C O R O L.

Retaining the above ſymbols; and putting N^{ii} for the value of two equal joint lives whoſe complements are n; N^{iii} for three ſuch joint lives; N^{iv} for four ſuch, *&c.*

Since $N = P - \frac{1-p}{n}Q$ by queſt. 56.

$$N^{ii} = P - \frac{2}{n}Q + \frac{1-p}{nn}R \qquad 65.$$

$$N^{iii} = P - \frac{3}{n}Q + \frac{3}{nn}R - \frac{1-p}{n^3}S \qquad 72.$$

$$N^{iv} = P - \frac{4}{n}Q + \frac{6}{nn}R - \frac{4}{n^3}S + \frac{1-p}{n^4}T \qquad 75.$$

$$\text{Therefore}\left\{\begin{array}{l} N^{v} = P - \frac{5}{n}Q + \frac{10}{nn}R - \frac{10}{n^3}S + \frac{5}{n^4}T - \frac{1-p}{n^5}V: \\ N^{vi} = P - \frac{6}{n}Q + \frac{15}{nn}R - \frac{20}{n^3}S + \frac{15}{n^4}T - \frac{6}{n^5}V + \frac{1-p}{n^6}W. \end{array}\right.$$

And the value of an annuity, on m equal joint lives, will be

$$P - \frac{m}{n}Q + \frac{m \cdot \overline{m-1}}{n \cdot 2n}R - \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{n \cdot 2n \cdot 3n}S$$

$$\left(+ \frac{m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \overline{m-3}}{n \cdot 2n \cdot 3n \cdot 4n}T \; (m) \pm \frac{1-p}{n^m}Z.\right.$$

In which expreſſion, Z ſignifies the $\overline{m+1}$ term in the ſeries of factors P, Q, R, S, &c. whoſe values are found in queſtions 15, 16, 17, 18, &c. and the laſt term (*viz.* that which is multiplied by the factor Z) will be affirmative, when the term immediately preceding is negative; and negative, when the term immediately preceding is affirmative.

If the value of an annuity on 4, 5, 6, &c. unequal joint lives ſhould be required; add their ages together, and take the 4th, 5th, 6th, &c. part of the ſum (if an integer) or the next leſſer integer thereto, for a mean age; with which, find the value of the annuity as above; which will be ſufficiently near the truth.

The reaſons of proceeding no farther, in the computation of unequal lives, are as follow:

Firſt, Becauſe queſtions of that kind ſeldom occur in practice, there being very few tenures, or leaſes on lives, wherein the number exceeds three.

Secondly, The computation derived from the principle, of the decrements of life being equal (which ſeems to be the only principle that can be applied thereto) would become exceſſive tedious; as would even the approximation, if it ſhould (which is doubted) turn out to be ſhorter than the other: 'Tis true, if the values of the ſingle lives be given, Mr. *De Moivre*'s approximation may be uſed with tolerable readineſs: And the manner of continuing thereof, to any number of joint lives is ſufficiently evident. But this ſeems to differ more from the truth, the greater the number of lives is; indeed, when a computation is required to be made, from the bills of mortality of *London*, it may be uſed with greater probability of exactneſs.

Thirdly, The greater the number of unequal lives are, the nearer will the reſult be found by the mean age; as might (if it needs a proof) be very eaſily made evident.

This method, of finding a mean age, may be uſeful, even in finding the values of two and three lives; when the ages given are not too far diſtant.

QUES-

QUESTION LXXVI.

Suppoſing the decrements of life to be equal, it is required to find the value of an annuity, upon the longeſt of two lives, whoſe reſpective complements of life are m and n, m being the greater number?

SOLUTION.

Since (by the preceding) the probabilities of the continuance of thoſe lives for one year are reſpectively $\frac{n-1}{n}$ and $\frac{m-1}{m}$; Therefore the probabilities of their ſeverally failing, will be $1-\frac{n-1}{n}$, and $1-\frac{m-1}{m}$ (by corol. to queſt. 26.) and conſequently the probability of their both failing, in that time $\overline{1-\frac{n-1}{n}} \times \overline{1-\frac{m-1}{m}}$, which probability, being ſubtracted from unity, will leave the probability of there being one of them (at leaſt) exiſting at the year's end:

But $\overline{1-\frac{n-1}{n}} \times \overline{1-\frac{m-1}{m}} = 1-\frac{n-1}{n}-\frac{m-1}{m}+\frac{\overline{n-1} \times \overline{m-1}}{nm}$ which, taken from unity, leaves $\frac{n-1}{n}+\frac{m-1}{m}-\frac{\overline{n-1} \times \overline{m-1}}{nm}$, the firſt payment of the annuity.

Again, ſince the probabilities of the given lives ſeverally failing in the ſecond year, are reſpectively $1-\frac{n-2}{n}$ and $1-\frac{m-2}{m}$; therefore the probability of their both failing will be $\overline{1-\frac{n-2}{n}}\times\overline{1-\frac{m-2}{m}}$; which being taken from unity, will leave $\frac{n-2}{n}+\frac{m-2}{m}-\frac{\overline{n-2}\times\overline{m-2}}{nm}$ the ſecond payment of the annuity.

By continuing the ſame kind of proceſs, for the third, fourth, &c. years, and finding their preſent values, it will appear, that the required annuity may be expreſſed in the following manner.

$$
\begin{array}{llll}
+\frac{n-1}{nr} & +\frac{m-1}{mr} & -\frac{\overline{n-1}\times\overline{m-1}}{nmr} \\
+\frac{n-2}{nr^2} & +\frac{m-2}{mr^2} & -\frac{\overline{n-2}\times\overline{m-2}}{nmr^2} \\
+\frac{n-3}{nr^3} & +\frac{m-3}{mr^3} & -\frac{\overline{n-3}\times\overline{m-3}}{nmr^3} \\
+\frac{n-4}{nr^4} & +\frac{m-4}{mr^4} & -\frac{\overline{n-4}\times\overline{m-4}}{nmr^4} \\
\quad \&c. & \quad \&c, & \quad \&c.
\end{array}
$$

But $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}$, &c. is the value of an annuity, on the ſingle life whoſe complement is n (by queſt. 56);

And $\frac{m-1}{mr}+\frac{m-2}{mr^2}+\frac{m-3}{mr^3}$, &c. is the value of an annuity, on the ſingle life whoſe complement is m;

Alſo $\frac{\overline{n-1}\times\overline{m-1}}{nmr}+\frac{\overline{n-}\times\overline{m-2}}{nmr^2}+\frac{\overline{n-3}\times\overline{m-3}}{nmr^3}$, &c.

is the value of an annuity upon the joint lives of the two perſons; and therefore,

If, from the ſum of the values of the ſingle lives of two perſons, the value of their joint lives be taken; the remainder will be the value of an annuity on the longeſt of them.

The above rule is the readieſt that can be deſired, if the values of the ſingle lives, and that of the joint lives, are known; but when they are not, it requires the ſolution of three queſtions, previous to this proceſs; or even if the values of the ſingle lives be known, the finding the value of the joint lives is laborious enough; proceed we, therefore, to find another rule, by the help of the general ſolutions of thoſe queſtions.

The value of the two ſingle lives will be, ſeverally,

$$\left.\begin{array}{ll} & P+Q\times\overline{-\frac{1}{n}+\frac{1}{nr^{n}}} \\ \text{And} & P+Q\times\overline{-\frac{1}{m}+\frac{1}{mr^{m}}} \end{array}\right\}\text{ by queſt. 56,}$$

And the value of the two joint lives will be

$$P+Q\times\overline{-\frac{1}{n}-\frac{1}{m}+\frac{1}{nr^{n}}-\frac{1}{mr^{n}}}+R\times\overline{\frac{1}{mn}-\frac{r}{nmr^{n}}},$$ by queſtion 64.

Whence it will appear, that when the expreſſion of the value of the joint lives, is taken from the ſum of the expreſſions of the values of the two ſingle lives, there will remain

$$P+Q\times\overline{\frac{1}{mr^m}+\frac{1}{mr^n}}-R\times\overline{\frac{1}{mn}-\frac{1}{nmr^n}},$$

for the value of the longeſt of the two lives;

Which expreſſion (reſtoring rPP for Q; $\overline{r+1}\times rP^3$ for R; and p for $\frac{1}{r^n}$) will become

$$P+\overline{\frac{1}{mr^m}+\frac{p}{m}}\times rPP-\overline{\frac{1}{nm}-\frac{p}{nm}}\times\overline{r+1}\times rP^3;$$

$$\text{Or } P+\overline{p+\frac{1}{r^m}}\times\frac{rPP}{m}-\frac{\overline{1-p}\times\overline{r+1}\times rP^3}{nm};$$

$$\text{That is } P-\overline{\frac{\overline{1-p}\times\overline{r+1}\times P}{n}-\overline{p+\frac{1}{r^m}}}\times\frac{rPP}{m}.$$

EXAMPLE.

What is the value of an annuity, for the longeſt of two lives, whoſe complements are 43 and 32, allowing compound intereſt at 4 *per Cent.*

Performed by the firſt Rule.

The value of a ſingle life, whoſe complement is 43, per example 1. queſt. 68.	12,683
The value of a ſingle life, whoſe complement is 54, per example 2, ditto	10,478
Their ſum	23,161
The value of an annuity, for the two joint lives, per example 1. queſt. 76.	8,276
Remains the anſwer.	14,885

The ſame performed by the laſt Rule.

Here $P = \left(\frac{1}{r-1} =\right) 25$; $p = 0{,}285058$; $\frac{1}{r^m} = 0{,}185168$;

$n = 32$; $m = 43$; and $\overline{r+1} = 2{,}04$; $1-p = 0{,}714942$;

$\frac{\overline{1-p} \times \overline{r+1} \times P}{n} = 1{,}139439$; And $rPP = 650$;

$p + \frac{1}{r^m} = 0{,}470226$;

Then the remainder $0{,}669213 \times \frac{650}{43} = 10{,}116$:

And $25 - 10{,}116 = 14{,}884$ the anſwer.

QUESTION LXXVII.

It is required to approximate to the value of the longeſt of two lives, ſuppoſing the decrements of life to be equal, and having the values of the ſingle lives given?

SOLUTION.

Let the value of the life whoſe complement n is the leſſer number, be denoted by N, and the value of the life whoſe complement is m, by M;

Then the value of the joint lives will be

$N - \overline{N - \frac{n-1}{6r}} \times \frac{n+1}{2m}$ by ſcholium to queſtion 64.

Therefore from $M+N$ (the ſum of the ſingle lives)

Take $N-\overline{N-\frac{n-1}{6r}}\times\frac{n+1}{2m}$.

And the remainder $M+\overline{N-\frac{n-1}{6r}}\times\frac{n+1}{2m}$,

will be the value of the annuity, for the longeſt of the two lives.

EXAMPLE.

The values of the two ſingle lives, of the ages 43 and 54; being 12,683, and 10,478; their complements 43, and 32; and the rate of intereſt 4 *per Cent.* what is the value of an annuity on the longeſt of thoſe two lives?

In example 1. ſcholium to queſt. 64. the value of the expreſſion,

$\overline{N-\frac{n-1}{6r}}\times\frac{n+1}{2m}$ was found to be 2,114;

To which add $M=12,683$;

The ſum 14,797, will be the value of the annuity required, which differs from the true value above found, only by being 0,087 too little.

The Rule in words at length, will ſtand as follows:

The Rule to find the preſent value of an annuity, to continue during the life of the longeſt liver of two perſons of given ages, allowing compound intereſt at a given rate, ſuppoſing

ſuppoſing the decrements of life to be equal, and the values of both the ſingle lives to be known.

Let the number of years, which each of the perſons want of 86, be called the complements of life; and let the ſum of one pound, and its intereſt for one year, be called the rate.

From the leſſer complement ſubtract one, and divide the remainder by ſix times the rate, or find this quotient in table the laſt.

From the value of the ſingle life of the oldeſt perſon, ſubtract the above-found quotient, and multiply the remainder, by the leſſer complement more one.

Divide the laſt found product, by twice the greater complement; and add the quotient to the value of the youngeſt life, ſo ſhall the ſum be the value of the annuity required.

EXAMPLE.

What is the value of an annuity of one pound, to continue during the life of the longeſt liver, of two perſons of the reſpective ages of 43, and 54, allowing compound intereſt at 4 per Cent. per Annum; the value of the life of 54 years being 10,478, and the value of the life of 43 years being 12,683:

Here (86—43=) 43, is the greater complement;
And (86—54=) 32, is the leſſer complement;
Alſo (1+,04=) 1,04, is the rate:

Now if (32— 1=) 31, be divided by (1,04×6=) 6,24, the quotient will be 4, 970.

From 10,478, take 4,970, and there will remain 5,508; which multiplied by (32+1=) 33 will produce 181,764.

If the laſt product (181,764) be divided by (43×2=) 86, the quotient will be 2,114.

Which quotient, being added to 12,683, gives 14,797; for the value of the annuity required.

N.B. In this, and the following caſes; the approximations, given by Mr. *De Moivre*, muſt (if uſed) be always applyed to the firſt given rules (*viz.* thoſe in which the values of the longeſt lives, and reverſions, are directed to be found, by adding, or ſubtracting, the values of the ſingle or joint lives) becauſe the expreſſions of thoſe approximations will not admit of being added, and ſubtracted, in the manner of the above.

QUESTION LXXVIII.

Suppoſing the decrements of life to be equal, it is required to find the value of an annuity, upon the longeſt of two equal lives, whoſe complement is n?

SOLUTION.

This may be deduced from the ſolution of queſt. 76; only by writing n for m, and p for $\frac{1}{r^m}$; as follows

$$P - \frac{\overline{\overline{1-p} \times \overline{r+1} \times P}}{n} - 2p \times \frac{rPP}{n}$$

Or (if deduced from the firſt expreſſion there given)

$$P + Q \times \frac{2p}{n} - R \times \frac{1-p}{nn}.$$

EX-

EXAMPLE.

What is the value of the longeſt of two equal lives each of 48 years; allowing compound intereſt at 4 *per Cent.*

Here $n=38$; $r=1,04$; $p=0,225285$; $P=25$.

$\overline{1-p}\times\overline{r+1}\times P=39,510475$, and $\frac{39,510475}{38}=1,039749$:

$PPr=650$; $2p=0,450570$;

$1,039749-0,450570=0,589179$;

And $0,589179\times\frac{25\times25\times1,04}{38}=10,07806$:

Laſtly $25-10,07806=14,92194$ the value of the annuity required.

QUESTION LXXIX.

'Tis required to approximate to the value of the longeſt of two equal lives, if the value of the ſingle life be known?

SOLUTION.

From the value of the two equal lives $2N$;

Take the value of the equal joint lives (by queſt. 65.) $\Big\}\ \frac{\overline{n-1}\times N}{2n}+\frac{n-1}{12r}$;

And the value of the longeſt life will be ——. $\Big\}\ 2N-\frac{\overline{n-1}\times N}{2n}-\frac{n-1}{12r}$.

Or $\frac{3n+1}{2n}N-\frac{n-1}{12r}$;

That is $\overline{\frac{\overline{3n+1}\times N}{n} - \frac{n-1}{6r}} \times \frac{1}{2}$.

Which, in words at length, follows.

The Rule, for finding the value of an annuity, upon the longeſt of two equal lives having the value of the ſingle life given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which the given life wants of 86, *be called the complement of life, and let the ſum of* 1 l. *and its intereſt for one year, be called the rate.*

To three times the complement add one; multiply the ſum by the value of the ſingle life; and divide that product by the complement.

From the complement ſubtract one, and divide the remainder by ſix times the rate; or find this quotient in table the laſt.

Then ſhall half the difference of thoſe two quotients be the value of the annuity required.

If the example given in the laſt queſt. be here propounded; then the value of the ſingle life will be 11,748; (86—48=) 38 will be the complement of life; and 1,04 the rate.

Then (3×38+1=) 115, being multiplied by 11,748; produces 1351,02; which being divided by 38 gives 35.553 for the quotient.

Again (38—1=) 37, being divided by (6×1,04=) 624, will give 5 929 for the quotient.

And, half the difference of thoſe quotients (*viz.* $\frac{35.553 - 5\ 929}{2}$ =) 14,812 will be the value of the annuity required.

QUESTION LXXX.

Suppoſing the decrements of life to be equal; it is required to find the value of an annuity, upon the longeſt of three lives (that is to continue as long as either of them is in being) whoſe reſpective complements are t, m and n (t being $\sqsubset m$, and $m \sqsubset n$)?

SOLUTION.

As in queſt. 76. $1-\frac{n-1}{n}$; $1-\frac{m-1}{m}$; and $t\frac{t-1}{t}$, will expreſs the probabilities of the given live's, ſeverally, failing in the firſt year, and therefore the probability of the failing of all of them the firſt year will be

$$\overline{1-\frac{n-1}{n}} \times \overline{1-\frac{m-1}{m}} \times \overline{1-\frac{t-1}{t}};$$ which being ſubtracted from unity, will leave the probability of their being one of them (at leaſt) exiſting at the year's end; which is the firſt payment of the annuity.

By continuing the proceſs, expanding the expreſſions by multiplication, ſubtracting the products from unity, and finding the preſent values of the remainders, it will appear, that the value of the required annuity will be expreſſed by the following ſeven ſeries.

$$\frac{n-1}{nr}+\frac{m-1}{mr}+\frac{t-1}{tr}-\frac{\overline{n-1}\times\overline{m-1}}{nmr}-\frac{\overline{n-1}\times\overline{t-1}}{ntr}$$

$$\frac{n-2}{nr^2}+\frac{m-2}{mr^2}+\frac{t-2}{tr^2}-\frac{\overline{n-2}\times\overline{m-2}}{nmr^2}-\frac{\overline{n-2}\times\overline{t-2}}{ntrr}$$

$$\frac{n-3}{nr^3}+\frac{m-3}{mr^3}+\frac{t-3}{tr^3}-\frac{\overline{n-3}\times\overline{m-3}}{nmr^3}-\frac{\overline{n-3}\times\overline{t-3}}{ntr^3}$$

&c. &c. &c. &c. &c.

$$\left(-\frac{\overline{m-1}\times\overline{t-1}}{mtr}+\frac{\overline{n-1}\times\overline{m-1}\times\overline{t-1}}{nmtr}\right.$$

$$\left(-\frac{\overline{m-2}\times\overline{t-2}}{mtr^2}+\frac{\overline{n-2}\times\overline{m-2}\times\overline{t-2}}{nmtr^2}\right.$$

$$\left(-\frac{\overline{m-3}\times\overline{t-3}}{mtr^3}+\frac{\overline{n-3}\times\overline{m-3}\times\overline{t-3}}{nmtr^3}\right.$$

&c. &c.

Now $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}$, &c. is the value of that ſingle life, whoſe complement is n;

And $\frac{m-1}{mr}+\frac{m-2}{mr^2}+\frac{m-3}{mr^3}$, &c. is the value of that ſingle life, whoſe complement is m;

Alſo $\frac{t-1}{tr}+\frac{t-2}{tr^2}+\frac{t-3}{tr^3}$, is the value of that ſingle life, whoſe complement is t:

Again $\frac{\overline{n-1}\times\overline{m-1}}{nmr}+\frac{\overline{n-2}\times\overline{m-2}}{nmr^2}+\frac{\overline{n-3}\times\overline{m-3}}{nmr^3}$, &c. is the value of thoſe two joint lives, whoſe complements are n and m;

$\frac{\overline{n-1}\times\overline{t-1}}{ntr}+\frac{\overline{n-2}\times\overline{t-2}}{ntr^2}+\frac{\overline{n-3}\times\overline{t-3}}{ntr^3}$, &c. is the value of thoſe two joint lives, whoſe complements are n and t;

And $\frac{\overline{m-1}\times\overline{t-1}}{mtr}+\frac{\overline{m-2}\times\overline{t-2}}{mtr^2}+\frac{\overline{m-3}\times\overline{t-3}}{mtr^3}$, &c. is the value of thoſe two joint lives, whoſe complements are m and t:

Laſtly $\frac{\overline{n-1}\times\overline{m-1}\times\overline{t-1}}{nmtr}+\frac{\overline{n-2}\times\overline{m-2}\times\overline{t-2}}{nmtr^2}$, &c. is the value of the three joint lives.

Hence may be deduced the following rule, for finding the value of the longeſt of three lives, *viz.*

To the ſum of the values of the three ſingle lives, add the value of the three joint lives; and from their ſum, ſubtract the three values of the joint lives, taken two and two, and the remainder will be the value of an annuity for the longeſt of the three lives.

EXAMPLE.

If the lives are of the reſpective ages of 43, 54, and 66; Then

The value of a ſingle life of $\left\{\begin{matrix}43\\54\\66\end{matrix}\right\}$ is $\left\{\begin{matrix}12,683\\10,478\\7,333\end{matrix}\right\}$ by queſt. 56.

Sum 30,494

The value of the 3 joint lives is } 5,152 by queſt. 69.

Their ſum 35,646

Then

Then from that ſum		35,646
Take	The value of 2 joint lives of 43&54=8,277	
	And of 43&66=6,272	
	And of 54&66=5,907	
	Found per queſt. 64	20,456
Remains the value of an annuity on the longeſt of the three lives		15,190

But as the above rule preſuppoſes the ſolution of ſeven queſtions, four of which require long operations; it will be worth while to find a ſolution independent of any of them: In order to which it will be neceſſary to add and and ſubtract the literal ſolutions, according to the form of the above numeral proceſs.

Let therefore the values of the annuities on the ſingle lives whoſe complements are n, m, and t, be denoted by N, M and F: The value of the joint lives whoſe complements are, n and m, by $\overline{NM}$; n and t, by $\overline{MF}$; and m and t, by $\overline{NF}$: And the value of the annuity on the three joint lives by $\overline{NMF}$.

Then $N+M+F-\overline{NM}-\overline{NF}-\overline{FM}+\overline{NMF}$ will be the value of the annuity required.

Now $N=P+Q\times\overline{-\frac{1}{n}+\frac{1}{nr^n}}$, by queſt. 56.

$$M=P+Q\times\overline{-\frac{1}{m}+\frac{1}{mr^m}},$$

And $F=P+Q\times\overline{-\frac{1}{t}+\frac{1}{tr^t}}$;

Whence $M+N+F$ will readily appear to be $3P+Q\times\overline{-\frac{1}{n}-\frac{1}{m}-\frac{1}{t}+\frac{1}{nr^n}+\frac{1}{mr^m}+\frac{1}{tr^t}}$.

Alſo $\overline{NM} = P + Q \times \overline{-\frac{1}{n} - \frac{1}{m} + \frac{1}{nr^n} - \frac{1}{mr^n}} + R \times \overline{\frac{1}{nm} - \frac{1}{nmr^n}}$ (by queſtion 64.)

$\overline{NF} = P + Q \times \overline{-\frac{1}{n} - \frac{1}{t} + \frac{1}{nr^n} - \frac{1}{tr^n}} + R \times \overline{\frac{1}{nt}} - \overline{\frac{1}{ntr^n}},$

And $\overline{MF} = P + Q \times \overline{-\frac{1}{m} - \frac{1}{t} + \frac{1}{mr^m} - \frac{1}{tr^m}} + R \times \overline{\frac{1}{mt}} - \overline{\frac{1}{mtr^m}};$

Whence the ſum of $\overline{NM}$, $\overline{NF}$, $\overline{MF}$ will be

$$3P + Q \times \overline{-\frac{2}{n} - \frac{2}{m} - \frac{2}{t} + \frac{2}{nr^n} + \frac{1}{mr^m} - \frac{1}{mr^n} - \frac{1}{tr^n} - \frac{1}{tr^m}} +$$

$$\left(R \times \overline{\frac{1}{nm} + \frac{1}{nt} + \frac{1}{mt} - \frac{1}{nmr^n} - \frac{1}{ntr^n} - \frac{1}{mtr^m}};\right.$$

Which being ſubtracted from $N+M+F$ will leave

$$\mathcal{Q}\times\overline{\frac{1}{a}+\frac{1}{m}+\frac{1}{t}-\frac{1}{nr^n}+\frac{1}{mr^n}+\frac{1}{tr^t}+\frac{1}{tr^n}+\frac{1}{tr^m}+}$$

$$\left(R\times\overline{-\frac{1}{nm}-\frac{1}{nt}-\frac{1}{mt}+\frac{1}{nmtr^n}+\frac{1}{ntr^n}+\frac{1}{mtr^m}};\right.$$

Again $\overline{NMF}=P+\mathcal{Q}\times\overline{-\frac{1}{t}-\frac{1}{m}-\frac{1}{n}+\frac{1}{tr^t}-\frac{1}{mr^n}+\frac{1}{nr^n}+\frac{n}{mtr^n}+}$

$$\left(R\times\overline{\frac{1}{mt}+\frac{1}{nt}+\frac{1}{nm}-\frac{1}{ntr^n}-\frac{1}{nmr^n}+\frac{2}{mtr^n}}-S\times\frac{1-p}{nmt}\right.\text{(by queſt. 69)}$$

Which added to the beforegoing remainder, will make,

$$P+\mathcal{Q}\times\overline{\frac{1}{tr^t}+\frac{1}{ttm}+\frac{n}{mtr^n}}+R\times\overline{\frac{2}{mtr^n}+\frac{1}{mtr^m}}-S\times\frac{1-p}{nmt},$$

The value of the annuity required;

Which expreſſion (if we reſtore rPP for Q; $\overline{r+1}\times rP^3$ for R; $\overline{\overline{r+2}^2\times r-3r}\times P^4$ for S; and p for $\frac{1}{rn}$) is

$$\overline{P+\frac{1}{r^t}+\frac{1}{r^m}+\frac{np}{m}}\times\frac{rPP}{t}$$

$$\left(+\overline{2p+\frac{1}{r^m}}\times\frac{\overline{r+1}\times rP^3}{mt}-\frac{\overline{1-p}\times\overline{\overline{r+2}^2\times r-3r}\times P^4}{nmt};\right.$$

And making $\frac{rPP}{t}=L$) will become

$$P+L\times\left\{\begin{array}{l}+\overline{\frac{n}{m}\times p+\frac{1}{r^t}+\frac{1}{r^m}}\\ +\overline{2p+\frac{1}{r^m}}\times\frac{\overline{r+1}\times P}{m}\\ -\frac{\overline{1-p}\times\overline{\overline{r+2}^2-3}\times P^2}{nm}\end{array}\right.,$$

the value of the annuity required.

Now if we will perform the operation of the above given example by this rule, then $n=20$; $m=32$; $t=43$; $p=0{,}456387$; $\frac{1}{r^m}=0{,}285058$; $\frac{1}{r^t}=0{,}185168$; & $P=25$: $\frac{rPP}{t}=L=15{,}11628$, and $\frac{n}{m}=\left(\frac{20}{32}=\frac{5}{8}=\right)$, 625;

$0{,}456387\times\frac{20}{32}=0{,}285242$;

$0{,}285242+0{,}285058+0{,}185168=0{,}755468$.

And $0{,}456387\times 2+0{,}285058=1{,}197832$.

Again $1,197832 \times \frac{2,04\times25}{32} = 1,90905$;

$r+2=3,04$; and $3,04\times3,04-3=6,2416$;

$1-0,456387 = 0,543613$;

$\frac{0,543613\times6,2416\times25\times25}{20\times32} = 3,313491$;

And $0,755468+1.90905-3,313491=-0,648973$:

Alſo $15,11628 \times -0,648973 = -9,81005$:

Laſtly $25-9,81005 = 15,18995$ the value of the annuity required.

QUESTION LXXXI.

Suppoſing the decrements of life to be equal, it is required to approximate to the value of an annuity on the longeſt of three lives, having their ages, and the value of each ſingle life given, and allowing compound intereſt at a given rate?

SOLUTION.

If the ſame ſymbols be retained as before,

$$\left.\begin{aligned}\text{Then } \overline{NM} &= N-\frac{\overline{n+1}\times N}{2m}+\frac{n-1}{6r}\times\frac{n+1}{2m}\\ \overline{NF} &= N-\frac{\overline{n+1}\times N}{2t}+\frac{n-1}{6r}\times\frac{n+1}{2t}\\ \overline{MF} &= M-\frac{\overline{m+1}\times M}{2t}+\frac{m-1}{6r}\times\frac{n+1}{2t}\end{aligned}\right\}\text{by ſchol. to queſt. 64.}$$

Alſo $\overline{MNF}=N\times\overline{1-\frac{n+1}{2m}}\times\overline{1-\frac{n+1}{2t}}$ $\left.\begin{array}{l} \\ \left(+\frac{n-1}{2r}\times\frac{n+1}{2m}\times\frac{\overline{m+t}\times 2-\overline{n+3}}{6t}\right.\end{array}\right\}$ by ſchol. to queſt. 69.

The firſt term of which laſt expreſſion, being expanded by multiplication, becomes

$$N-\frac{\overline{n+1}\times N}{2m}\quad\frac{\overline{n+1}\times N}{2t}+\frac{n+1}{2m}\times\frac{n+1}{2t}N;$$

which being ſubtracted from the value of $\overline{NM}+\overline{NF}$ will leave $N+\frac{n+1}{6r}\times\frac{n+1}{2m}+\frac{n-1}{6r}\times\frac{n+1}{2t}-\frac{n+1}{2m}\times\frac{n+1}{2t}N$

Or $N+\overline{\frac{1}{m}+\frac{1}{t}}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}-\frac{\overline{n+1}\times\overline{n+1}}{4mt}N$:

Now, for the ſecond term of the value of $\overline{NMF}$ $\left(\text{viz. }\frac{n-1}{2r}\times\frac{n+1}{2m}\times\frac{\overline{m+t}\times 2-\overline{n+3}}{6t}\right)$ may be wrote $\frac{\overline{n-1}\times\overline{n+1}}{12r}\times\frac{\overline{m+t}\times 2-\overline{n+3}}{2mt}$; which being ſubtracted from the ſecond term of the above remainder $\left(\text{viz }\overline{\frac{1}{m}+\frac{1}{t}}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}\right)$ there will remain

$$\overline{\frac{1}{m}+\frac{1}{t}-\frac{\overline{m+t}\times 2-\overline{n+3}}{2mt}}\times\frac{\overline{n-1}\times\overline{n+1}}{12r},$$

Or $\frac{n+3}{2mt}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}$:

And then, the value of $\overline{NM}+\overline{NF}-\overline{NMF}$ will be

$$N+\frac{n+3}{2tm}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}-\frac{\overline{n+1}\times\overline{n+1}}{4mt}N;$$

Therefore the value of $\overline{NM}+\overline{NF}+\overline{MF}-\overline{NMF}$ will be

$$N+\frac{n+3}{2mt}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}-\frac{\overline{n+1}^{2}\times N}{4mt}$$

$$\Big(+M-\frac{\overline{m+1}\times M}{2t}+\frac{\overline{m-1}\times\overline{m+1}}{12rt},$$

Or $N+M-\frac{\overline{n+1}^{2}\times N}{4mt}-\frac{\overline{m+1}\times M}{2t}+\frac{\overline{m-1}\times\overline{m+1}}{12rt}$

$$\Big(+\frac{\overline{n+3}}{2tm}\times\frac{\overline{n-1}\times\overline{n+1}}{12r};$$

Which being ſubtracted from $N+M+F$ will leave,

$$F+\frac{\overline{n+1}^{2}\times N}{4mt}+\frac{\overline{m+1}\times M}{2t}-\frac{\overline{m-1}\times\overline{m+1}}{12rt}$$

$\Big(-\frac{n+3}{2mt}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}$, the value of the annuity.

Now, ſince the above approximation to the value of the annuity is found by ſubtracting of three approximations (*viz.* thoſe to the 3 values of two joint lives) each of which exceeds the truth; and by the adding only of one approximation (*viz.* that to the value of three joint lives) which is alſo greater than the truth; it will follow that the value above found will be leſs than juſt:

And ſince $\Big(\frac{n+3}{2tm}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}=\Big)\frac{n-1}{6r}\times\frac{\overline{n+3}\times\overline{n+1}}{4mt}$ is a quantity, to be ſubtracted, in the above approximation; therefore if that quantity be leſſened, the approximation will be encreaſed.

But that quantity will not be leſſened (and conſequently the approximation not encreaſed) by $\frac{2}{12}$, if $\frac{n-1}{6r}\times\frac{\overline{n+1}^{2}}{4mt}$ be wrote in its ſtead: For $\frac{\overline{n-1}\times\overline{n+1}}{tm}\times\frac{n+3}{24r}$ may be wrote for the ſaid expreſſion; where $\left(\frac{\overline{n-1}\times\overline{n+1}}{tm}=\right)$ $\frac{nn-1}{tm}$ muſt be leſs than unity, becauſe both t and m are greater than n; and by writing $\frac{n+1}{24r}$ for $\frac{n+3}{24r}$, we leſſen that part of the expreſſion by $\left(\frac{2}{24r}=\right)$ $\frac{1}{12r}$; which, becauſe r is greater than unity, is leſs than $\frac{1}{12}$: And conſequently the whole will not be leſſened by $\frac{1}{12}$ of unity.

Putting therefore $\frac{n-1}{6r}\times\frac{\overline{n+1}^{2}}{4mt}$, for $\frac{n+3}{2mt}\times\frac{\overline{n-1}\times\overline{n+1}}{12r}$, the approximation to the value of the annuity will become

$$F+\frac{\overline{n+1}^{2}}{4mt}N+\frac{\overline{m+1}\times M}{2t}-\frac{\overline{m-1}\times\overline{m+1}}{12rt}-\frac{n-1}{6r}\times\frac{\overline{n+1}^{2}}{4mt},$$

Or $F+\frac{\overline{n+1}^{2}}{4mt}N-\frac{\overline{n+1}^{2}}{4mt}\times\frac{n-1}{6r}+M\times\frac{m+1}{2t}-\frac{m-1}{6r}\times\frac{m+1}{2t}$

That is $F+\overline{N-\frac{\overline{n-1}}{6r}}\times\frac{\overline{n+1}^{2}}{4mt}+\overline{M-\frac{m-1}{6r}}\times\frac{m+1}{2t}$,

Or $F+\frac{1}{2t}\times\overline{\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^{2}}{2m}+\overline{M-\frac{m-1}{6r}}\times\overline{m+1}}$

Which, expreſſed in words at length, follows.

The rule for finding the value of an annuity, on the longeſt of three unequal lives, of given ages; having the values of thoſe ſingle lives given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which each life wants of 86, be called their complements of life; and let 1 L, and its intereſt for one year, be called the Rate.

From the leaſt complement ſubtract one, and divide the remainder by ſix times the rate; or find this quotient in table the laſt; ſubtract this quotient from the value of the eldeſt life, reſerving the remainder.

To the leſſer complement add one, multiply the ſum by itſelf; and that product by the remainder above reſerved; dividing this laſt product by twice the middle complement, reſerving the quotient.

From the middle complement ſubtract one, and divide the remainder by ſix times the rate; or find the quotient by table the laſt; ſubtract the quotient from the value of the middle life, and multiply the remainder by the middle complement more one.

To the laſt found product, add the quotient above reſerved, and divide their ſum by twice the greateſt complement; to which quotient, add the value of the youngeſt life, and the ſum will be the value of the annuity required.

The operation of the example, given in queſt. 80, by this rule, may ſtand as follows.

Here the value of the youngeſt life will be 12,683
of the middle 10,478
of the eldeſt 7,333

And (86—43=) 43 will be the greateſt complement,
(86—54=) 32 middle
(86—66=) 20 leaſt

Alſo (1+,04=) 1,04, the rate.

Now, if (20—1=) 19 be divided by 6×1,04=6,24, the quotient will be 3,045; which, ſubtracted from 7,333, leaves 4,288, for the remainder to be reſerved.

And, if (20+1=) 21 be multiplied by 21, it produces 441; and 441 multiplied by the above reſerved remainder 4,288, produces 1891,008; which being divided by (2×32=) 64 will give 29,547, for the quotient to be reſerved.

Alſo, if (32-1=) 31 be divided by 6,24, the quotient will be 4,968; which, ſubtracted from 10,478, leaves 5,510; and this multiplied by (32+1=) 33, produces 181,830.

Again, the laſt product 181,830, being added to 29,547, the reſerved quotient, makes 211,377; which, divided by (43×2=) 86, quotes 2,458; and the ſum of 12 683 & 2,458 (*viz.* 15,141) will be the value of the annuity required.

QUESTION LXXXII.

Suppoſing the decrements of life to be equal, it is required to find the value of an annuity, upon the longeſt of three lives; two of which are of equal ages, and younger than the third.

The ſolution of this queſtion may be deduced from queſt. 80, by writing m for t; and $\frac{1}{r^m}$, for—$\frac{1}{r^t}$; whence the value of the annuity required will be

$$P+\frac{rP^2}{m}\times\left\{\begin{array}{l}\frac{n}{m}\times p+\frac{2}{r^m}+\overline{2p+\frac{1}{r^m}}+\frac{\overline{r+1}\times P}{m}\\ -\frac{\overline{1-p}\times\overline{\overline{r+2}^2-3}\times P^2}{nm}\end{array}\right.$$

For example; Let the given equal ages be 54, and the elder life 66; alſo let compound intereſt be allowed at 4 *per Cent.*

Then $m=32$; $n=20$; $r=1{,}04$; $p=0{,}456387$; $P=25$, $1-p=0{,}543613$; $\overline{r+2}^2-3=6{,}2416$; $nm=640$; $\frac{1}{r^m}=0{,}285058$; $\frac{n}{m}=0{,}625$; $0{,}625\times0{,}456387=0{,}285241$;
$2\times0{,}456387=0{,}912774$; and $0{,}912774+0285058=1{,}197832$;

Again, $1{,}197832\times\frac{2{,}04\times25}{32}=1{,}90905$; $\frac{PPr}{m}=20{,}3125$
$\frac{0{,}543613\times6{,}2416\times25\times25}{20\times32}=3{,}313491$;

And $0,285241+0,570116+1,90905=2,76441$;
Alſo $2,76441-3,31349=-0,54908$;
$-0,54908\times20,3125=-11,1532$;

Laſtly, $25-11,1532=13,8468$, is the value of the annuity required.

SCHOLIUM.

If the approximation, given in the laſt queſtion, be applied to the value of the annuity, by writing m for t and M for F, it will become

$M+\frac{1}{2m}\times\overline{\overline{N-\frac{n-1}{6r}}\times\overline{\frac{n+1}{2m}}^{2}+\overline{M-\frac{m-1}{6r}}\times\overline{m+1}}$. The expreſſion of which, in words at length, will differ from the former, only, in writing, greater complement, inſtead of greateſt and middle complement; and younger life, for middle and youngeſt life.

QUESTION LXXXIII.

Suppoſing the decrements of life to be equal, it is required to find the value of an annuity upon the longeſt of three lives, two of which are of equal ages, and elder than the third.

The ſolution of this queſtion may alſo be deduced from queſt. 80, by writing n for m, and p for $\frac{1}{,m}$; whence the value of the annuity will be,

$$* \quad P+\frac{rP^2}{t}\times\left\{\begin{array}{l}\frac{n}{n}\times p+\frac{1}{rt}+p+3p\times\frac{\overline{r+1}\times P}{n}\\ -\frac{\overline{1-p}\times\overline{\overline{r+2}^2-3\times P^2}}{nn},\end{array}\right.$$

$$\text{Or } P+\frac{rP^2}{t}\left\{\begin{array}{l}2p+\frac{1}{rt}+3p\times\frac{\overline{r+1}\times P}{n}\\ -\frac{\overline{1-p}\times\overline{\overline{r+2}^2-3\times P^2}}{nn};\end{array}\right.\text{ That is}$$

$$P+\frac{rP^2}{t}\times\left\{\begin{array}{l}\frac{1}{rt}+\overline{2+\frac{\overline{r+1}\times 3P}{n}}\times p\\ -\frac{\overline{1-p}\times\overline{\overline{r+2}^2-3\times P}}{nn}.\end{array}\right.$$

For example; let the given equal ages be 66; and the younger 43; and let compound interest be allowed at 4 *per Cent.*

Here $t=43$; $p=0,456387$; $1-p=0,543613$; $P=25$; $n=20$; $\frac{1}{rt}=0,185168$; $\frac{rPP}{t}=\frac{650}{43}=15,11628$; $r+1=2,04$; $2,04\times 3=6,12$; $6,12\times\frac{25}{20}=7,65$; $\overline{7,65+2}\times 0,456387=4,404134$; $\overline{r+2}^2-3=6,2416$;

And $\frac{0,543613\times 6,2416\times 25\times 25}{20\times 20}=5,301586$;

Now $0,185168+4,404134-5,301586=-0,712284$;

And $15,11628\times -0,712284=-10,76708$;

Laſtly, 25−10,76708=14,23292, the value of the annuity required.

QUESTION LXXXIV.

Suppoſing the decrements of life to be equal, it is required to approximate to the value of an annuity upon the longeſt of three lives, two of which are equal, and elder than the third.

SOLUTION.

This may be readily ſolved, by writing *N* for *M*, and *n* for *m*, in the reſult of queſt. 81; whence the value of the annuity will be

$$P+\frac{1}{2t}\times\overline{\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^2}{2n}+\overline{N-\frac{n-1}{6r}}\times\overline{n+1}},$$

$$\text{Or } P+\frac{1}{2t}\times\overline{N-\frac{n-1}{6r}}\times\overline{\frac{\overline{n+1}^2}{2n}+n+1};$$

$$\text{Now } \frac{\overline{n+1}^2}{2n}+\overline{n+1}=\left(\overline{\frac{n+1}{2n}+1}\times\overline{n+1}=\right)\frac{\overline{3n+1}\times\overline{n+1}}{2n}$$

Th. $P+\frac{\overline{3n+1}\times\overline{n+1}}{4nt}\times\overline{N-\frac{n-1}{6r}}$ will be the value of the annuity.

Which may be expreſſed in words, at length, as follows.

The rule for finding the value of an annuity, on the longeſt of three lives, two of which are equal and elder than the third, having the values of the ſingle lives given, allowing intereſt

intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which each life wants of 86, *be called their complements of life, and let one pound, and its intereſt for one year, be called the rate.*

From the leſſer complement ſubtract one, and divide the remainder by ſix times the rate, or find this quotient by table the laſt; ſubtract this quotient from the value of one of the elder lives, reſerving the remainder.

To three times the leſſer complement add one, and multiply the ſum by the ſame complement more one; *alſo multiply this product by the remainder above reſerved, and divide the quotient by four times the product of the two complements.*

To the laſt found quotient, add the value of the younger life, and the ſum will be the value of the annuity required.

The operation of the example, given in queſt. 83, may ſtand as follows.

Here the value of the younger life will be 12,683
elder 7,333
And (86—43=) 43 will be the greater complement,
(86—66=) 20 leſſer
Alſo (1+,04=) 1,04 the rate.
Now if (20—1 =) 19 be divided by (6×1,04=) 6,24, the quotient will be 3,045 ; which, ſubtracted from 7,333, leaves 4,288, for the remainder to be reſerved.

And if (3×20+1=) 61 be multiplied by (20+1=) 21, the product will be 1281 ; which, being alſo multiplied by 4,288, will produce 5492,928.

Now ($4 \times 43 \times 20 =$) 3440 is the product of four times the two complements; by which, if 5492,928 (the number above found) be divided, the quotient will be 1,597; to which adding 12,683 (the value of the younger life) the ſum, 14,280, will be the value of the annuity required.

QUESTION LXXXV.

Suppoſing the decrements of life to be equal, it is required to find the value of an annuity, upon the longeſt of three equal lives.

The ſolution of this may be alſo reduced from queſt. 80, by writing n for m and t; and p for $\frac{1}{r^m}$ and $\frac{1}{r^t}$; and then the value of the annuity will be

$$\frac{1}{r-1}+\frac{3p}{n}\times\frac{r}{\overline{r-1}^{2}}+\frac{3p}{nn}\times\frac{\overline{r+1}\times r}{\overline{r-1}^{3}}-\frac{1-p}{n^{3}}\times\frac{\overline{r+2}^{2}\times r-3r}{\overline{r-1}^{4}}, \text{ Or}$$

$$P+\frac{3p}{n}Q+\frac{3p}{nn}R-\frac{1-p}{n^{3}}S, \text{ that is}$$

$$P+\frac{rP^{2}}{n}\times\left\{\begin{array}{l}3p+3p\times\dfrac{\overline{r+1}\times P}{n}\\ -\dfrac{\overline{1-p}\times\overline{\overline{r+2}^{2}-3}\times PP}{nn}\end{array}\right.;\text{ that is}$$

$$P+\frac{rP^{2}}{n}\left\{\begin{array}{l}\overline{1+\dfrac{\overline{r+1}\times P}{n}}\times 3P\\ -\dfrac{\overline{1-p}\times\overline{\overline{r+2}^{2}-3}\times PP}{nn},\end{array}\right.$$

For example; let the three perſons be each 54 years old, and compound intereſt be allowed at 4 *l. per Cent.*

Then $32=n$; $0,285058=p$; $1,04=r$; $0,714942=1-p$; & $25=P$; $(r+1=)$ $2,04\times25=51,00$; and $\frac{51}{32}=1,59375$; alſo $3p=0,855174$; $0,855174\times2,59375=2,21807$; $\overline{r+2}^2-3=6,2416$; $6,2416\times0,714942=4,4624$; and $4,4624\times\frac{25\times25}{32\times32}=2,723649$;

Now $2,723649-2,218107\times\frac{25\times25\times1,04}{32}=10,2688$;

Whence $25-10,2688=14,7312$, the value of the annuity required.

SCHOLIUM I.

If we make uſe of the approximation in queſt. 84, by writing n for t and N for F, the value of the annuity will be

$N+\frac{\overline{3n+1}\times\overline{n+1}}{4nn}\times\overline{N-\frac{n-1}{6r}}$ which may be expreſſed in words in the ſame manner as that remembering that the lives and the complements are equal.

SCHOLIUM II.

If it were required to find the value of an annuity, to continue as long as any two (out of three perſons of given ages) ſhall be alive; the ſame might be inveſtigated as follows.

If their complements of life be repreſented by t, m, and n, as before.

Then the probability of the continuance, for one year, of the lives of

- all three will be $\frac{t-1}{t}\times\frac{m-1}{m}\times\frac{n-1}{n}$
- the two eldeſt $\frac{m-1}{m}\times\frac{n-1}{n}\times\overline{1-\frac{t-1}{t}}$
- the eldeſt and youngeſt $\frac{t-1}{t}\times\frac{n-1}{n}\times\overline{1-\frac{m-1}{m}}$
- the two youngeſt $\frac{t-1}{t}\times\frac{m-1}{m}\times\overline{1-\frac{n-1}{n}}$

Which kind of proceſs being continued, and the preſent worths being found, it will appear, that an annuity during the joint continuance of two lives out of three, will conſiſt of the three joint lives of the given perſons taken two and two, leſs twice the joint lives of all the three.

The actual addition and ſubtraction of the reſpective values of the unequal lives (as the reſult will not be often wanted) is left as an exerciſe to the reader; but is inſerted for equal lives, becauſe that operation will take up but little room; thus,

From three times the value of two equal joint lives, $\Big\}\; 3P-\frac{3\times2}{n}Q+\frac{3\times\overline{1-p}}{nn}R,$

Take twice the val. of three equal joint lives. $\Big\}\ 2P-\frac{2\times3}{n}Q+\frac{2\times3}{nn}R-\frac{2\times\overline{1-p}}{n^3}S;$

Remains the value of an annuity to continue as long as 2 of 3 lives. $\Big\}\ P-\frac{3\times\overline{1+p}}{nn}R+\frac{2\times\overline{1-p}}{n^3}S.$

QUESTION LXXXVI.

Suppoſing the decrements of life to be equal, it is requir'd to find the value of an annuity, upon the longeſt of four equal lives.

SOLUTION.

Let n be the complement of life, then will $1-\frac{n-1}{n}$ expreſs the probability of either of them failing in the firſt year; and therefore the probability of the failing of all of them in that year will be

$$\left(\overline{1-\frac{n-1}{n}}^{\,4}=\right)1-4\times\frac{n-1}{n}+6\times\frac{\overline{n-1}}{nn}-4\times\frac{\overline{n-1}^{\,3}}{n^3}+\frac{\overline{n-1}}{n^4}$$

which, being ſubtracted from unity, will leave,

$4\times\frac{n-1}{n}-6\times\frac{\overline{n-1}^{\,2}}{n^2}+4\times\frac{\overline{n-1}^{\,3}}{n^3}-\frac{\overline{n-1}^{\,4}}{n^4}$, for the probability, that one (at leaſt) of them will be alive at the year's end; therefore the firſt payment will be worth $4\times\frac{n-1}{nr}-6\times\frac{\overline{n-1}^{\,2}}{n^2r}+4\times\frac{\overline{n-1}^{\,3}}{n^3r}-\frac{\overline{n-1}^{\,4}}{n^4r}$: And by continuing the proceſs, the ſecond, third, &c. years payments will be worth

4×

$$4\times\frac{n-2}{nr^2}-6\times\frac{\overline{n-2}^2}{n^2r^2}+4\times\frac{\overline{n-2}^3}{n^3r^2}-\frac{\overline{n-2}^4}{n^4r^2},$$

$$4\times\frac{n-3}{nr^2}-6\times\frac{\overline{n-3}^2}{n^2r^3}+4\times\frac{\overline{n-3}^3}{n^3r^3}-\frac{\overline{n-3}^4}{n^4r^3}$$

&c.

Now $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}$ (*n*) is the value of the ſingle life, whoſe complement is *n*;

$\frac{\overline{n-1}^2}{nnr}+\frac{\overline{n-2}^2}{nnr^2}+\frac{\overline{n-3}^2}{nnr^3}$ (*n*) is the value of two equal joint lives, whoſe complements are *n*;

$\frac{\overline{n-1}^3}{n^3r}+\frac{\overline{n-2}^3}{n^3r^2}+\frac{\overline{n-3}^3}{n^3r^3}$ (*n*) is the value of three equal joint lives, whoſe complements are *n*;

And $\frac{\overline{n-1}^4}{n^4r}+\frac{\overline{n-2}^4}{n^4r^2}+\frac{\overline{n-3}^4}{n^4r^3}$ (*n*) is the value of four equal joint lives, whoſe complements are *n*.

Therefore, if, from four times the value of the ſingle life, ſix times the value of two equal joint lives be taken; and then, to the remainder, four times the value of three equal joint lives be added; and laſtly, from the ſum, the value of the four equal joint lives be taken; then ſhall the laſt remainder be the value of the longeſt of the four equal lives.

Now, if we aſſume the ſame ſymbols as in queſt. 20. then,

$$4N = 4P - \frac{4-4p}{n}Q;$$

$$-6N^{ii} = -6P + \frac{12}{n}Q - \frac{6-6p}{nn}R;$$

$$+4N^{iii} = 4P - \frac{12}{n}Q + \frac{12}{nn}R - \frac{4-4p}{n^3}S;$$

$$-N^{iv} = -P + \frac{4}{n}Q - \frac{6}{nn}R + \frac{4}{n^3}S - \frac{1-p}{n^4}T:$$

Therefore, the value of the longeſt of the four equal lives will be

$$P + \frac{4p}{n}Q + \frac{6p}{nn}R + \frac{4p}{n^3}S - \frac{1-p}{n^4}T$$

COROL.

Retaining the above ſymbols, and writing $L.N^{ii}$ for the value of the longeſt of two equal lives, whoſe complements are n; $L.N^{iii}$ for the value of the longeſt of three ſuch lives; $L.N^{iv}$, for the value of four ſuch, &c.

Since $N = P - \frac{1-p}{n}Q;$ by queſt. 56

$L \cdot N^{ii} = P + \frac{2p}{n}Q - \frac{1-p}{nn}R;$ 78

$L \cdot N^{iii} = P + \frac{3p}{n}Q + \frac{3p}{nn}R - \frac{1-p}{n^3}S;$ 85

$L \cdot N^{iv} = P + \frac{4p}{n}Q + \frac{6p}{nn}R + \frac{4p}{n^3}S - \frac{1-p}{n^4}T;$ 186

Therefore $\left\{\begin{array}{l} L\cdot N^{v}=P+\frac{5p}{n}Q+\frac{10p}{nn}R+\frac{10p}{n^3}S+\frac{5p}{n^4}T-\frac{1-p}{n^5}V. \\ L\cdot N^{vi}=P+\frac{6p}{n}Q+\frac{15p}{nn}R+\frac{20p}{n^3}S+\frac{15p}{n^4}T+\frac{6p}{n^5}V-\frac{1-p}{n^6}W; \end{array}\right.$

And the value of an annuity, on the longeſt of m equal lives will be

$$P+\frac{mp}{n}Q+\frac{mp\cdot\overline{m-1}}{n\cdot 2n}R+\frac{mp\cdot\overline{m-1}\cdot\overline{m-2}}{n\cdot 2n\cdot 3n}S$$

$$\left(+\frac{mp\cdot\overline{m-1}\cdot\overline{m-2}\cdot\overline{m-3}}{n\cdot 2n\cdot 3n\cdot 4n}T\,(m)-\frac{1-p}{n^m}Z.\right.$$

Where Z ſignifies the $\overline{m+1}$ term of the ſcale of factors, P, Q, R, S, &c. whoſe values are found in queſt. 15, 16, 17, &c.

QUESTION

QUESTION LXXXVII.

Suppoſing the decrements of life to be equal; it is required to find the value of an annuity, to continue as long as any three (out of four equal lives) ſhall be in being.

SOLUTION.

If n be the complement of life, then will $\frac{\overline{n-1}^4}{n^4}$ be the probability of the continuance of all the four lives for one year; $\left(\frac{\overline{n-1}^3}{n^3}\times\overline{1-\frac{n-1}{n}}=\right)\frac{\overline{n-1}^3}{n^3}-\frac{\overline{n-1}^4}{n^4}$ will be the probability of the failing of ſome one (particularly named) and of the continuance of the other three for one year; and becauſe there are four perſons, each of which may be ſo particularly named, therefore that probability muſt be taken four times.

Th. $\left(\frac{\overline{n-1}^4}{n^4}+\overline{\frac{\overline{n-1}^3}{n^3}-\frac{\overline{n-1}^4}{n^4}}\times 4=\right)\frac{\overline{n-1}^3\times 4}{n^3}-\frac{\overline{n-1}^4\times 3}{n^4}$ will be the value required.

That is, if from four times the value of three equal joint lives, three times the value of the four equal joint lives be taken, the remainder will be the value of an annuity to continue as long as any three of them ſhall be alive.

Now four times the value of three equal joint lives is $\left.\right\} 4P - \frac{4\times 3}{n}Q + \frac{4\times 3}{nn}R - \frac{4\times\overline{1-p}}{n^3}S$;

And three times the value of four equal joint lives $\left.\right\} 3P - \frac{3\times 4}{n}Q + \frac{3\times 6}{nn}R - \frac{3\times 4}{n^3}S + \frac{3\times\overline{1-p}}{n^4}T$;

Remains the value of three out of four equal lives $\left.\right\} P - \frac{6}{nn}R + \frac{2\times 4+4p}{n^3}S - \frac{3\times\overline{1-p}}{n^4}T$.

COROL.

From the two laſt proceſſes we may find the value of an annuity, to continue as long as $\overline{m-1}$ out of m equal lives (whoſe complements are n) ſhall be in being; thus, put N^{m} for the value of m equal joint lives; and N^{m-1} for the value of $\overline{m-1}$ equal joint lives.

From $m\times N^{m-1} = mP - \frac{m\times\overline{m-1}}{n}Q + \frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{n\cdot 2n}R$

$$\left(\text{&c.}\ \pm\frac{m\times\overline{1-p}}{n^{m-1}}Y,\right.$$

Take $\overline{m-1}\times N^{m} = \overline{m-1}\times P - \frac{\overline{m-1}\times m}{n}Q + \frac{\overline{m-1}\cdot m\cdot\overline{m-1}}{n\cdot 2n}$

$$\left(R\ \text{&c.}\pm\frac{\overline{m-1}\times m}{n^{m-1}}Y \pm \frac{\overline{m-1}\times\overline{1-p}}{n^{m}}Z,\right.$$

Remains the value of the annuity required.

$$= P - \frac{m\cdot\overline{m-1}}{n\cdot 2n}R + \frac{2m\cdot\overline{m-1}.\overline{m-2}}{n\cdot 2n\cdot 3n}S - \frac{3m\cdot\overline{m-1}\cdot\overline{m-2}\cdot\overline{m-3}}{n\cdot 2n\cdot 3n\cdot 4n},$$

$$\left(\text{&c.}\mp\frac{\overline{m-2+p}\times m}{n^{m-1}}Y \pm \frac{\overline{m-1}\times\overline{1-p}}{n^{m}}Z.\right.$$

COROL.

By reaſoning in a ſimilar manner, the value of an annuity, to continue as long as any two (out of four lives) ſhall be in being, may be found.

$$\text{For}\ \left.\begin{array}{l}\frac{\overline{n-1}^{4}}{n^{4}}\\[4pt] \frac{\overline{n-1}^{3}}{n^{3}}\times\overline{1-\frac{n-1}{n}}\\[4pt] \frac{\overline{n-1}^{2}}{n^{2}}\times\overline{1-\frac{n-1}{n}}^{2}\end{array}\right\}\begin{array}{l}\text{will be the pro-}\\ \text{bability of the}\\ \text{continuance, for}\\ \text{one year, of}\end{array}\left\{\begin{array}{l}\text{all four lives,}\\[4pt] \text{any 3 of them}\\[4pt] \text{any 2 of them}\end{array}\right.$$

But becauſe there are four combinations of three, and ſix combinations of two, in four; therefore the ſecond probability muſt be taken four times, and the third ſix times.

And the probability of receiving the firſt year's rent will be

$$\frac{\overline{n-1}^4}{n^4}+\frac{4\times\overline{n-1}^3}{n^3}\times\overline{1-\frac{n-1}{n}}+\frac{6\times\overline{n-1}^2}{n^2}\times\overline{1-\frac{n-1}{n}}^2$$

But $\frac{\overline{n-1}^3}{n^3}\times\overline{1-\frac{n-1}{n}}=\frac{\overline{n-1}^3}{n^3}-\frac{\overline{n-1}^4}{n^4}$,

And $\frac{\overline{n-1}^2}{n^2}\times\overline{1-\frac{n-1}{n}}^2=\frac{\overline{n-1}^2}{n^2}-\frac{2\times\overline{n-1}^3}{n^3}+\frac{\overline{n-1}^4}{n^4}$;

Whence $\frac{6\times\overline{n-1}^2}{n^2}-\frac{12\times\overline{n-1}^3}{n^3}+\frac{6\times\overline{n-1}^4}{n^4}$,

$$+\frac{4\times\overline{n-1}^3}{n^3}-\frac{4\times\overline{n-1}^4}{n^4},$$

$$+\frac{\overline{n-1}^4}{n^4},$$

That is $\frac{6\times\overline{n-1}^2}{n^2}-\frac{8\times\overline{n-1}^3}{n^3}+\frac{3\times\overline{n-1}^4}{n^4}$ will be the probability of receiving the firſt year's rent; and the value of the annuity may be found by adding ſix times the value of two joint lives to three times that of four joint lives; and ſubtracting eight times the value of three joint lives from the ſum.

Now ſix times the value of two joint lives is $\left.\right\}\ 6P-\frac{12}{n}Q+\frac{6\times\overline{1-p}}{nn}R$, three times that of four joint lives $\left.\right\}$ $3P-\frac{12}{n}Q+\frac{18}{nn}R-\frac{12}{n^3}S+\frac{3\times\overline{1\cdot p}}{n^4}T$;

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their ſum $9P-\frac{24}{n}Q+\frac{24-6p}{nn}R-\frac{12}{n^3}S+\frac{3\times\overline{1\cdot p}}{n^4}T$:

eight times that of 3 joint lives $\left.\right\}$ $8P-\frac{24}{n}Q+\frac{24}{nn}R-\frac{8\times\overline{1-p}}{n^3}S$,

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Remains $P-\frac{6p}{nn}R-\frac{4+8p}{n^3}S+\frac{3\times\overline{1-p}}{n^4}T$, the value of the annuity.

And thus we might proceed to find the value of an annuity, to continue as long as any *k* lives (out of *m* lives) ſhall be in being; but enough has been ſaid to enable the reader to perform this, if he has leiſure and inclination; therefore we proceed to things of more uſe.

QUESTION LXXXVIII.

It is required to find the value of the reverſion of an eſtate, in fee ſimple, after a ſingle life of a given age, allowing compound intereſt at a given rate,

SOLUTION.

From the value of the perpetuity (*viz.* $\frac{1}{r-1}$) ſubtract the value of the given life, found *per* queſt. 68, or 73 (which, if the decrements of life are equal, will be $\frac{r}{r-1}-\frac{\overline{1-p}\times r}{n\times r-1}$) and the remainder, (in that caſe $\frac{\overline{1-p}\times r}{n\times\overline{r-1}^2}$) will be the value of the reverſion required; which (if we put $\frac{r}{\overline{r-1}^2}=Q$) will become $\frac{1-p}{n}Q$.

The above, expreſſed in words at length, follows:

The rule, for finding the value of the reverſion of an eſtate of 1 l. per annum *in fee ſimple, after a life of a given age, allowing compound intereſt, at a given rate, and ſuppoſing the decrements to be equal.*

Let the number of years, which the given age wants of 86, *be called the complement of life.*

Seek in the tables for the preſent worth of 1 *l. due at the end of the complement of life; ſubtract the number ſo found from unity; multiply the remainder by the number, which (in the table annexed to queſt.* 20) *ſtands on a line with the given rate, under the letter* Q; *and divide the product by the complement of life, ſo ſhall the quotient be the value of the reverſion required.*

E X A M P L E.

What is the value of the reverſion of an eſtate in fee ſimple, of 1 *l. per annum*, after the life of a perſon of 54 years of age, allowing compound intereſt at 4 *per Cent.*?

Here (86-54=) 32. will be the complement of life,

0,285058 will be the preſent worth of 1 *l.* due at the end of 32 years.

And 650 will be the tabular number under Q.

Then if ($1-0{,}285058=$) 0,714942 be multiplied by 650, the product will be 464,712; which, divided by 32, will quote 14,522 the value of the reverſion.

QUESTION LXXXIX.

If it is required to find the preſent worth of 1 *l.* which is not to be received until the death of a perſon of a given age, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

SOLUTION.

One pound, or any other ſum of money, may be conceived as the preſent worth of an eſtate in fee ſimple, equal to the intereſt of that ſum for one year; and conſequently the reverſion of a fee ſimple equal to that intereſt, after the given life, will be the value required.

Now (retaining the ſame ſymbols as in queſt. 88) the intereſt of 1 *l.* for 1 year will be $r-1$; & if $\frac{\overline{1-p}\times r}{n\times\overline{r-1}^{2}}$ be multiplied by $r-1$, the product $\frac{\overline{1-p}\times r}{n\times r-1}$ or $\frac{\overline{1-p}\times r}{n}P$, will be the value required.

Which, expreſſed in words at length, follows:

The rule for finding the preſent worth of 1 *l. due on the deceaſe of a perſon of a given age, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.*

Let the number of years, which the given age wants of 86, *be called the complement of life; and let one pound, and its intereſt for one year, be called the rate.*

Seek in the tables for the preſent worth of one pound due at the end of the complement of life; ſubtract the number ſo found from unity; multiply the remainder by the rate, and that product by the number, which (in the table annexed to queſtion 20) ſtands on a line with the given rate, under the letter P.

Divide the laſt product by the complement of life, ſo ſhall the quotient be the preſent worth required.

EXAMPLE.

A, who is heir to a conſiderable fortune upon the death of *B*, aged 54 years, would borrow a ſum of money to be repaid, at that time, by him or his ſucceſſors; how much ought he to receive now, for every pound that is then to be paid, allowing compound intereſt at four *per Cent.*?

Here (86—54=) 32 will be the complement of life,
0,285058 will be the preſent worth of one pound due at the end of 32 years,
And 25 will be the tabular number under *P*.

Then if (1—0,285058=) 0,714942 be multiplied by 1,04 (the rate) it will produce 0,74354; and if this be multiplied by 25, the product will be 18,5885; which, being divided by 32, will give 0,5809, for the preſent worth required.

COROL.

In the ſame manner, the preſent worth of one pound, to be received at the expiration of any number of lives, may be found from the ſolutions of queſtions 89, 90, &c. by multiplying their reſult by the intereſt of one pound for one year.

Or, if the value of the given life or lives, be known; then, multiply it by the intereſt of one pound for one year; and ſubtract the product from unity, and the remainder will be the preſent worth required.

In the above example, the given life is worth 10,478; which, being multiplied by 0,04, produces ,41912; and this, being ſubtracted from unity, leaves ,58088, as above.

SCHOLIUM.

To obviate any doubt, that may ariſe in the reader's mind concerning the truth of the principle on which this queſtion has been ſolved; it is thought proper to inſert the following proceſs; which is performed in the manner frequently uſed by Mr. *De Moivre.*

Let N repreſent the value of an annuity on the given life; and let us ſuppoſe it to be equal to an annuity for a number of years certain, ſuppoſe m; then it will be evident, that the propoſed ſum will not be due till the expiration of thoſe m years; and conſequently one pound then due will be worth but $\frac{1}{r^m}$; by queſtion 144, part 2, and vol. I.

Now the preſent value of an annuity of one pound to continue m years, is $\frac{r^m-1}{r^m \times \overline{r-1}}$ (by queſtion 149, part 2, vol. I.) which expreſſion is ſuppoſed to be equal to the value of the life.

That is $\frac{r^m-1}{r^m \times \overline{r-1}} = N$; or $r^m - 1 = \overline{r-1} \times Nr^m$;

Th. $r^m - \overline{r-1} \times Nr^m = 1$; or $r^m = \frac{1}{1 - \overline{r-1} \times N}$;

Conſequently $\frac{1}{r^m} = \left(\frac{1-\overline{r-1}\times N}{1} = \right) \overrightarrow{1-r-1\times}^{N}$:
which is the laſt given rule.

But $N = \frac{1}{r-1} - \frac{\overline{1-p}\times r}{n\times\overline{r-1}^{2}}$ by queſt. 56;

Th. $\overline{r-1}\times N = 1 - \frac{\overline{1-p\times r}}{n\times\overline{r-1}}$;

And $1-\overline{r-1}\times N = \frac{\overline{1-p}\times r}{n\times\overline{r-1}}$ which is the ſame expreſſion as before.

In like manner, if it were required to know what ſum ought to be paid, on the deceaſe of a perſon of a given age, in conſideration of one pound now received, the ſame will appear to be $\frac{n\times\overline{r-1}}{\overline{1-p}\times r}$.

For the amount of one pound at compound intereſt for the time m, is $r^m = \frac{1}{1-\overline{r-1}\times N}$, by the above proceſs:

Alſo $1-\overline{r-1}\times N = \frac{\overline{1-p}\times r}{n\times\overline{r-1}}$; therefore $r^m = \frac{n\times\overline{r-1}}{\overline{1-p}\times r} = \frac{n}{\overline{1-p}\times rP}$.

The rule, in words at length, given in page 274, will ſerve for this purpoſe; if, inſtead of the laſt paragraph, you read thus

Divide the complement of life by the laſt found product, and the quotient will be the ſum, which ought to be paid at the expiration of the given life.

Note, the ſymbol (N) may be above expounded by the value of any number of lives; when ſuch value enters the queſtion, inſtead of the value of a ſingle life.

QUESTION XC.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an eſtate in fee ſimple, after the longeſt liver of two perſons of given ages?

SCHOLIUM.

If the ſame ſymbols be retained as in queſt. 76, then if from P, the value of the perpetuity, be ſubtracted the value of an annuity for the longeſt of the two lives, the remainder, *viz.*

$$\frac{rPP}{m}\times\overline{\frac{\overline{1-p}\times\overline{r+1}\times P}{n}-\overline{p+\frac{1}{rm}}},$$ will be the value required.

Or, by the approximation in queſt. 77, the value will be

$$P-M-N-\overline{\frac{n-1}{6r}\times\frac{n+1}{2m}}.$$

The numerical proceſs, and rules in words at length, are omitted in this, and ſome of the following queſtions, which will but rarely occur in practiſe; becauſe they differ very little from the proceſſes, and rules given in the queſtions, in this and thoſe following, quoted: To inſtance in this, if the reſult of the approximation in queſt. 77, be taken from the number, which (in the table annexed to queſt. 20) ſtands on a line with the given rate, under the letter *P*; the remainder will be the value of the reverſion.

Alſo, if the given ages are equal, then the value of the reverſion will (*per Cent.* 78) be

$$R\times\frac{1-p}{nn}-Q\times\frac{2p}{n}, \text{ Or } R\times\overline{\frac{1-p}{n}-2pQ}\times\frac{1}{n}.$$

And the approximation thereto, will (*per* queſt. 79) be

$$P-\tfrac{1}{2}\times\overline{\frac{\overline{3n+1}\times N}{n}-\frac{n-1}{6r}}.$$

QUESTION XCI.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an eſtate, in fee ſimple, after the longeſt liver of three perſons of given ages.

SOLUTION.

If the ſame ſymbols be retained as in queſt. 80, then, if from P, the value of the perpetuity, be ſubtracted the value of an annuity on the longeſt of the three given lives, the remainder will be the value of the reverſion required, *viz.*

$$\frac{rP^2}{t}\times\left\{\begin{array}{l}+\dfrac{\overline{1-p}\times\overline{\overline{r+2}^2-3}\times P^2}{nm}\\[2ex]-\overline{\dfrac{n}{m}p+\dfrac{1}{r^t}+\dfrac{1}{r^m}}\\[2ex]-\overline{2p+\dfrac{1}{r^m}}\times\dfrac{\overline{r+1}\times P}{m}\end{array}\right.$$

And the approximation thereto will (*per* queſt. 81) be

$$P-F-\frac{1}{2t}\times\overline{\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^2}{2m}+\overline{M-\frac{m-1}{6r}}\times\overline{m+1}}$$

Alſo, if the three ages be equal, then the value of the reverſion will (*per* queſt. 85) be

$$\frac{1-p}{n^3}S-\frac{3p}{nn}R-\frac{3p}{n}Q.$$

And the approximation thereto will (*per* ſcholium to the ſame queſt) be

$$P-N-\frac{\overline{3n+1}\times\overline{n+1}}{4nn}\times\overline{N-\frac{n-1}{6r}}.$$

COROL.

Since the reverſion of an eſtate, in fee ſimple, after the longeſt of

$$\left.\begin{array}{l}1\\2\\3\end{array}\right\}\text{equal lives, is}\begin{cases}\dfrac{1-p}{n}Q;\\[2ex]\dfrac{1-p}{nn}R-\dfrac{2p}{n}Q;\\[2ex]\dfrac{1-p}{n^3}S-\dfrac{3p}{nn}R-\dfrac{3p}{n}Q;\end{cases}$$

Therefore, the reverſion of an eſtate, in fee ſimple, after the longeſt of m lives, will be

$$-\frac{mp}{n}Q-\frac{m\cdot\overline{m-1}\cdot p}{1\cdot 2\cdot nn}R-\frac{m\cdot\overline{m-1}\,.\overline{m-2}\cdot p}{1\cdot 2\cdot 2\cdot n^3}S(m-1)+\frac{1-p}{n^m}Z.$$

Where Z denotes the $\overline{m+1}$ factor in the ſcale, P, Q, R, &c. (queſt: 20).

SCHOLIUM.

The queſtions relating to the reverſion of an eſtate, in fee ſimple, after two or more unequal joint lives, are omitted, becauſe it is ſuppoſed that the reſolution thereof will hardly ever occur in practice; And becauſe the principle, on which their ſolution depends, is the ſame with the former,

mer, *viz.* the taking the value of the lives from the perpetuity.

Whence, and from corol. to queſt 75, it will appear, that the value of the reverſion of an eſtate, in fee ſimple, after

$$\left.\begin{array}{l}2\\[1ex]3\\[1ex]4\end{array}\right\}\begin{array}{l}\text{joint lives}\\\text{will be}\end{array}\left\{\begin{array}{l}\frac{2}{n}Q-\frac{1-p}{nn}R\\[1ex]\frac{3}{n}Q-\frac{3}{nn}R+\frac{1-p}{n^3}S\\[1ex]\frac{4}{n}Q-\frac{6}{nn}R+\frac{4}{n^3}S-\frac{1-p}{n^4}T\end{array}\right.$$

&c.

And thererefore the value of the reverſion of an eſtate, in fee ſimple, after *m* joint lives, will be

$$\frac{m}{n}Q-\frac{m\cdot\overline{m-1}}{n\cdot 2n}R+\frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{n\cdot 2n\cdot 3n}S(m-1)\pm\frac{1-p}{n^4}Z$$

In which expreſſion, Z ſignifies the $m+1$ term of the ſeries of factors P, Q, R, &c· (ſee queſt. 20) and the laſt term will be affirmative, when the term immediately preceding is negative; and negative, when that is affirmative.

QUESTION XCII.

There is a leaſehold eſtate of one pound *per annum*, to continue *n* years; which *A* (whoſe complement of life is *m*) is to enjoy, if he lives ſo long; but if he dies before the expiration of the ſaid *n* years, then *B* and his heirs are to have the remainder thereof; what is the value of *B*'s intereſt therein, ſuppoſing the decrements of life to be equal?

SOLU-

SOLUTION.

From $\overline{1-\frac{1}{r^n}}\times\frac{1}{r-1}$ the value of the annuity for *n* years certain (queſt. 15), ſubtract

$$\overline{1-\frac{1}{r^n}}\times\frac{1}{r-1}+\frac{n}{mr^n\times\overline{r-1}}-\overline{1-\frac{1}{r}}\times\frac{r}{m\times\overline{r-1}^2}$$

the preſent worth of the annuity for *n* years, if *A* ſhall live ſo long (queſt. 57), and the remainder *viz.*

$$\overline{1-\frac{1}{r^n}}\times\frac{r}{m\times\overline{r-1}^2}-\frac{n}{mr^n\times\overline{r-1}}$$

; Or, (putting $P=\frac{1}{r-1}$, and $\mathcal{Q}=\frac{r}{\overline{r-1}^2}$) $\overline{1-\frac{1}{r^n}}\times\frac{\mathcal{Q}}{m}-\frac{nP}{mr^n}$,

That is $\frac{\mathcal{Q}}{m}-\frac{\mathcal{Q}}{mr^n}-\frac{nP}{mr^n}$, Or $\frac{\mathcal{Q}}{m}-\frac{\mathcal{Q}+nP}{mr^n}$;

Or (putting $\frac{1}{r^n}=p$) $\frac{\mathcal{Q}-\overline{\mathcal{Q}+nP}\times p}{m}$ will be the value required.

Which, expreſſed in words at length, follows:

The rule for finding the value of the reverſion of a leaſehold eſtate of one pound per annum, *of which a given number of years remain unexpired, after the deceaſe of a perſon of a given age; allowing compound intereſt at a given rate and ſuppoſing the decrements of life to be equal.*

Let the number of years, which the given age wants of 86, *be called the complement of life.*

Multiply that number which (in the table annexed to queſt. 20) *ſtands, on a line with the given rate, under the letter* P, *by the number of years unexpired in the leaſe; to the product add the number, ſtanding under the letter* $\mathcal{Q}$, *in the ſame table; and multiply the ſum, by the preſent worth of one pound due at the expiration of the leaſe.*

Subtract this laſt found product from the before-mentioned tabular number, ſtanding under the letter Q; and divide the remainder by the complement of life, ſo ſhall the quotient be the value of the reverſion required.

EXAMPLE.

A, who is 43 years of age, is poſſeſſed of a leaſehold eſtate of one pound *per ann.* (whereof 32 years are unexpired) which eſtate, if *A* ſhould die before that time, deſcends to *B* and his heirs: I demand the value of *B*'s intereſt in that eſtate, allowing compound intereſt at 4 *per Cent.*

Here (86−43=) 43 will be the complement of life,
The tabular number ſtanding under *P* will be 25;
That under Q 650;
The preſent worth of one pound, due at the end of 32 years, will be ,285058;
Now if 32, be multiplied by 25, the product will be 800; to which, if 650 be added, the ſum will be 1450; which, multiplied by ,285058, produces 413,334.
And if 413,334 be ſubtracted from 650, the remainder will be 236,666; which, being divided by 43, quotes 5,504, the value of the reverſion required.

QUESTION XCIII.

There is an eſtate of one pound *per annum* ſo entailed, that if *A*, a perſon of a given age, ſhould die before he arrives at a certain age alſo given; then the eſtate ſhall deſcend to *B* and his heirs; what is the value of *B*'s intereſt therein, ſuppoſing the decrements of life to be equal?

SOLU-

SOLUTION.

Subtract the years of *A*'s preſent age, from thoſe of the age to which he muſt attain before *B*'s expectation ceaſes, and call the remainder *n*; and let *m* be the complement of *A*'s life; then will *B*'s intereſt in the eſtate, for the firſt *n* years, be the difference between an annuity certain for thoſe years, and the like annnity ſubject to *A*'s life.

Now an annuity for *n* years certain is $\overline{1-\frac{1}{r^n}}\times\frac{1}{r-1}$ by queſt. 15; and the value of the like annuity ſubject to the life of *A* is (*per* queſt. 57) $\overline{1-\frac{1}{r^n}}\times\frac{1}{r-1}+\frac{n}{mr^n}\times\frac{1}{r-1}-\overline{1-\frac{1}{r}}\times\frac{r}{m\times\overline{r-1}^2}$ the difference of which is $\overline{1-\frac{1}{r^n}}\times\frac{r}{m\times\overline{r-1}^2}-\frac{n}{mr^n\times r-1}$

Alſo the probability of *A*'s living the ſaid *n* years will be $\frac{m-n}{m}$; and therefore the probability of his dying in that time will be $\left(1-\frac{m-n}{m}=\right)\frac{n}{m}$; which is the probability of *B*'s having the reverſion of the perpetuity, at the expiration of *n* years.

But the reverſion of the perpetuity after *n* years will be $\frac{1}{r^n\times\overline{r-1}}$ by queſt. 15; and therefore $\frac{n}{mr^n\times\overline{r-1}}$ will be *B*'s intereſt in that reverſion; which, being added to the above-found expreſſion of *B*'s intereſt in the firſt *n* years, will give $\overline{1-\frac{1}{r^n}}\times\frac{r}{m\times\overline{r-1}^2}$, Or $\overline{1-\frac{1}{r^n}}\times\frac{Q}{m}$

(by writing as before *Q* for $\frac{r}{\overline{r-1}^2}$) for the whole value of *B*'s intereſt.

COROL.

If the queſtion had propoſed to have found the value of this reverſion from tables of obſervations; then the value of an annuity for *n* years, if *A* ſhall live ſo long (which may be found by queſt. 60) being taken from the value of an annuity for *n* years certain; the difference will ſhew *B*'s intereſt for the firſt *n* years: alſo if *a* and *b* be, ſeverally, the numbers proportional to the living of *A*'s age, and at the end of *n* years, then $\frac{b_n}{a \times r^n \times \overline{r-1}}$ will be the remaining part of his intereſt for ever.

The above rule, expreſſed in words at length, follows:

The rule for finding the value of the reverſion of an eſtate of one pound per annum, *in fee ſimple, after the death of a minor (if that happen before he comes to be of age to poſſeſs it) allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.*

Let the number of years, which the minor wants of 86, be called his complement of life; and, which he wants of being at age, his complement of poſſeſſion.

From unity ſubtract the preſent worth of one pound due at the end of the complement of poſſeſſion; multiply the remainder by the number which (in the table annexed to queſt. 20) ſtands, on a line with the rate, under the letter Q; and divide the product by the complement of life, ſo ſhall the quotient be the value of the reverſion required.

EXAM-

EXAMPLE.

Suppoſe that *B* and his heirs are entitled to an eſtate of one pound *per annum*, if *A* (who is 10 years of age) ſhould die before he is 21; what is the reverſion worth, allowing compound intereſt at four *per Cent*.

Here (86—10=) 76 will be the complement of life,

And (21—10=) 11 that of poſſeſſion.

Alſo the preſent worth of one pound due at the end of 11 years is 0,6496;

And the number ſtanding, on a line with the rate, under *Q*, is 650.

Now if (1—0,6496=) 0,3504, be multiplied by 650, the product will be 227,76; which, being divided by 76, will quote 2,997, the value of the reverſion required.

QUESTION XCIV.

Suppoſing the the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the life of a perſon of a given age, after the deceaſe of the preſent poſſeſſor, whoſe age is alſo given?

SOLUTION.

Let the complement of the life of the preſent poſſeſſor be denoted by n, and that of the expectant in reverſion by m.

Then the probabilities of the poſſeſſor's dying, in the firſt, ſecond, third, &c. years, will be

$1-\frac{n-1}{n}$, $1-\frac{n-2}{n}$, $1-\frac{n-3}{n}$ &c.

And the probability of the expectant's living to the end of the firſt, ſecond, third, &c. years, will be

$\frac{m-1}{m}$, $\frac{m-2}{m}$, $\frac{m-3}{m}$ &c.

And ſince theſe two events are independent on each other, it will follow (from queſt. 28) that,

$\frac{m-1}{m}\times\overline{1-\frac{n-1}{n}}$; $\frac{m-2}{m}\times\overline{1-\frac{n-2}{n}}$ &c. will be the probabilities of the expectant's receiving of the firſt, ſecond, &c. yearly payment ;

Which probabilities, being taken as the values of th firſt, ſecond, &c. payments ; being expanded by multiplication, and their preſent values being ſound, will gi the value of the reverſion, *viz.*

$$\frac{m-1}{mr}-\frac{\overline{m-1}\times\overline{n-1}}{mnr}+\frac{m-2}{mr^2}-\frac{\overline{m-2}\times\overline{n-2}}{mnr^2}\text{ \&c.}$$

Now $\frac{m-1}{mr}+\frac{m-2}{mr^2}+\frac{m-3}{mr^3}$ &c. is the value of an annuity on the life of the expectant ;

And $\frac{\overline{m-1}\times\overline{n-1}}{mnr}+\frac{\overline{m-2}\times\overline{n-2}}{mnr^2}+\frac{\overline{m-3}\times\overline{n-3}}{mnr^3}$ &c. the value of an annuity, to continue during the joint li of the poſſeſſor and expectant.

Therefore, if from the value of the expectant's life, t value of the joint lives of the poſſeſſor and expectant be tak the remainder will be the value of the reverſion :

Or, if from the value of the longeſt of the two lives, life of the preſent poſſeſſor be taken ; the remainder will the value of the reverſion.

Now the value of the longeſt of the two lives (per queſt. 76) is $P+Q\times\overline{\frac{1}{mr^m}+\frac{1}{mr^n}}-R\times\frac{1-p}{mn}$;

And the value of the poſſeſſor's life (per queſt. 56) is $P+Q\times\overline{\frac{1}{n}+\frac{1}{nr^n}}$;

The difference of which will be

$Q\times\overline{\frac{1}{n}+\frac{1}{mr^m}+\frac{1}{mr^n}-\frac{1}{nr^n}}-R\times\frac{1-p}{mn}$, the value of the reverſion.

Which expreſſion (reſtoring rPP, for Q; $\overline{r+1}\times rP^3$, for R, and p for $\frac{1}{r^n}$) will become

$$rPP\times\overline{m+\frac{n}{r^m}+np-mp}\times\frac{1}{nm}-P^3r\times\overline{r+1}\times\frac{\overline{1-p}}{mn},$$

$$\text{Or } \frac{rPP}{nm}\times\overline{m+\frac{n}{r^m}-\overline{m-n}\times p-\overline{r+1}\times P\times\overline{1-p}}.$$

EXAMPLE.

Let the expectant be 43 years of age, and the poſſeſſor 54 years, allowing compound intereſt at four *per Cent.* Then $(86-43=)\ 43=m$; $(86-54=)\ 32=n$; $p=0{,}285058$; $\frac{1}{r^m}=0{,}185168$; $r=1{,}04$; $\overline{r+1}=2{,}04$ & $P=25$; $32\times0{,}185188=5{,}925376$; $\overline{m-n}\times p=3{,}135638$; $\overline{1-p}\times\overline{r+1}\times P=36{,}462050$:
Then $43+5{,}925378-3{,}135638-36{,}462050=9{,}327688$;
And $\frac{rPP}{nm}=\frac{650}{1376}$, Th. $\frac{650}{1376}\times9{,}327688=4{,}406$.

Which is the ſame anſwer, as will ariſe from the proceſs firſt above directed, &c.

From the value of the expectant's life, which, exam. 1, queſt. 56, is - - - - - - } 12,683,

Take the value of the joint lives, which, by exam. 1, queſt. 64, is - - - - - - } 8,277;

Remains the value of the reverſion - - - 4,406.

Or, if the approximation to the value of the joint lives be made uſe of, then

From the value of the expectant's life M,

Take the value of the joint lives, *viz.* $N - \overline{N - \frac{n-1}{6r}} \times \frac{n+1}{2m}$

The remainder, $M - N + \overline{N - \frac{n-1}{6r}} \times \frac{n+1}{2m}$, will be the value of the reverſion.

Which, in words at length, follows:

The rule for finding the value of the reverſion of an annuity, for a given life, after the failure of another given life, elder than the former; having the values of the ſingle lives given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which the given lives want of 86, *be called their complements; and let the ſum of one pound and its intereſt for one year, be called the rate.*

From the leſſer complement ſubtract one, and divide the remainder by ſix times the rate, or find this quotient in the laſt table.

Subtract this quotient from the value of the poſſeſſor's life; multiply the remainder by the leſſer complement more one; *and divide the product by twice the greater complement.*

Laſtly, from the value of the life of the expectant take the value of the life of the poſſeſſor; and to the remainder, add *the laſt found quotient; ſo ſhall the ſum be the value of the reverſion required.*

EXAMPLE.

B, who is 54 years of age, is poſſeſſed of an annuity of one pound, which, after his deceaſe, is to deſcend to *A*, aged 43 years, if he ſurvives *B*, for the remaining part of his life; what is the preſent value of *A*'s intereſt in the annuity, allowing compound intereſt at four *per Cent.*?

Here the value of *A*, the expectant's life, is 12,683,
And the value of *B*, the poſſeſſor's life, is 10,478;
Alſo (86−43=) 43 is the greater complement,
(86−54=) 32 is the leſſer complement;
And (1+,04=) 1,04 is the rate.

If (32−1=) 31 be divided by (6×1,04=)6,24, the quotient will be 4,970.

Then (10,478−4,970=) 5,508, being multiplied by (32+1=) 33, produces 181,764; which, divided by (43×2=) 86, quotes 2,114.

Laſtly, (12,683−10,478=) 2,205; and (2,205+2,114=) 4,319 will be the value of the reverſion required.

CASE 2. When the poſſeſſor is younger than the expectant, then the complement of the expectant's life $=n$, and the poſſeſſor's $=m$, the value of the poſſeſſor's life will be,

$$P+Q\times\overline{-\frac{1}{m}+\frac{1}{mr^{m}}};$$

Which, taken from the value of the longeſt of the two lives, will leave $Q\times\overline{\frac{1}{m}+\frac{1}{mr^{n}}}-R\times\frac{1-p}{mn}$, the value of the reverſion;

Which (writing p for $\frac{1}{,n}$, and rPP for Q, *&c.*) will become, $\overline{1+p-\frac{\overline{1-p}\times r+1\times P}{n}}\times\frac{PPr}{m}$.

For example, let the expectant be 54 years of age, and the poſſeſſor 43 years, allowing compound intereſt at four *per Cent.*

Then $m=43$; $n=32$; $p=285058$; $r=1{,}04$; and $P=25$; $\overline{1-p}\times\overline{r+1}\times P=36{,}46205$; and $\frac{36{,}46205}{32}=$ $1{,}139438$:

Now $(1{,}285058-1{,}139438=)\ 0{,}14562\times\frac{650}{43}=2{,}2012$ the value of the reverſion required.

Or, if the approximation to the value of the joint lives be made uſe of, then,

From the value of the expectant's life N,

Take the value of the joint lives, *viz.*

$$N-\overline{N-\frac{n-1}{6r}}\times\frac{n+1}{2m};$$

The remainder, $N-\overline{\frac{n-1}{6r}}\times\frac{n+1}{2m}$, will be the value of the reverſion required.

Which, expreſſed in words at length, follows:

The rule for finding the reverſion of an annuity, for a given life after the failure of another given life, younger than the former; having the value of the elder life given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which the given lives want of 86, be called their complements; and let the ſum of one pound, and its intereſt for one year, be called the rate.

Fr

From the leſſer complement ſubtract one, and divide the remainder by ſix times the rate, or find the quotient in the laſt table.

Subtract this quotient from the given value of the elder life; multiply the remainder by the leſſer complement more one; *and divide the product by twice the greater complement; ſo ſhall the quotient be the value of the annuity required.*

EXAMPLE.

A, who is 43 years of age, is poſſeſſed of an annuity of one pound, which, after his deceaſe, is to belong to *B*, who is 54 years old (if he ſurvives *A*) for the remaining part of his life; what is the preſent value of *B*'s intereſt in the annuity, allowing compound intereſt at four *per Cent.*?

Here the value of *B* the elder life is 10,478;

(86—43=) 43, will be the greater complement
(86—54=) 32, leſſer
And (1+,04=) 1,04, the rate

Now (32— 1=) 31, being divided by (6×1,04=) 6,24; the quotient will be 4,970.

Then (10,478—4,970=) 5,508 being multiplied by (32+1=) 33, produces 181,764; which, divided by (43×2=) 86, quotes 2,114, the value of the reverſion required.

CASE 3. When the poſſeſſor and expectant are of equal ages, then the value of the reverſion will be

$$\overline{\overline{1+p}-\frac{\overline{1-p}\times\overline{r+1}\times P}{n}}\times\frac{PPr}{n},\ \text{Or}\ \frac{1+p}{n}\underline{Q}-\frac{1-p}{nn}R:$$

And the approximation will be

$\overline{N}-\frac{n-1}{6r}\times\frac{n+1}{2n}$; as will appear by writing *n* for *m* in the expreſſions obtained in the laſt caſe.

The proceſs, being the ſame, in both methods, with that given in the example to the laſt caſe (excepting only that the complements are equal) an example would be ſuperfluous.

QUESTION XCV.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the joint lives of two perſons of given ages, after the deceaſe of the preſent poſſeſſor, whoſe age is alſo given.

SOLUTION.

Let the complement of the life of the preſent poſſeſſor be denoted by n, and thoſe of the expectants by t and m;

Then the probability of the poſſeſſor's dying, in the firſt, ſecond, or third, &c. years, will be

$$1-\frac{n-1}{n},\ 1-\frac{n-2}{n},\ 1-\frac{n-3}{n}\ \&c.$$

And the probability of the joint continuance of the two expectants lives will, for the firſt, ſecond, third, &c. years, be

$$\frac{\overline{m-1}\times\overline{t-1}}{mt},\ \frac{\overline{m-2}\times\overline{t-2}}{mt},\ \frac{\overline{m-3}\times\overline{t-3}}{mt}\ \&c.$$

Which probabilities, being ſeverally multiplied into the correſponding probability of the poſſeſſor's dying, will give the values of the firſt, ſecond, &c. payments of the reverſion, *viz.*

m—

$$\frac{\overline{m-1}\times\overline{t-1}}{mt}-\frac{\overline{m-1}\times\overline{t-1}\times\overline{n-1}}{mtn}+\frac{\overline{m-2}\times\overline{t-2}}{mt}-\left(\frac{\overline{m-2}\times\overline{t-2}\times\overline{n-2}}{mtn},\ \&c.\right.$$

the preſent values of which payments will be

$$\frac{\overline{m-1}\times\overline{t-1}}{mtr}-\frac{\overline{m-1}\times\overline{t-1}\times\overline{n-1}}{mtnr}+\frac{\overline{m-2}\times\overline{t-2}}{mtr^2}-\left(\frac{\overline{m-2}\times\overline{t-2}\times\overline{n-2}}{mtnr^2},\ \&c.\right.$$

Now $\frac{\overline{m-1}\times\overline{t-1}}{mtr}+\frac{\overline{m-2}\times\overline{t-2}}{mtr^2}+\frac{\overline{m-3}\times\overline{t-3}}{mtr^3}$, &c. is the value of an annuity on the joint lives of the two expectants, and $\frac{\overline{m-1}\times\overline{t-1}\times\overline{n-1}}{mtnr}+\frac{\overline{m-2}\times\overline{t-2}\times\overline{n-2}}{mtnr^2}$ &c. is the value of an annuity on the joint lives of the poſſeſſor and two expectants.

Therefore; if, from the value of the joint lives of the two expectants, be taken the value of the three joint lives of the poſſeſſor and two expectants, the remainder will be the reverſion of the two joint lives after one.

The application of the above expreſſions of the values of joint lives to this rule admits of eight caſes.

CASE I. When the poſſeſſor is older than either of the expectants, their ages being unequal.

Then the value of an annuity on the joint lives of the two expectants will be

$$P+\overline{\frac{1}{mr^m}-\frac{1}{tr^m}-\frac{1}{m}-\frac{1}{t}}\times Q+\overline{\frac{1}{mt}-\frac{1}{mtr^m}}\times R;$$

and the value of any annuity on three joint lives will be

$$\overline{P+\frac{1}{nr^n}+\frac{n}{mtr^n}-\frac{1}{tr^n}-\frac{1}{mr^n}-\frac{1}{t}-\frac{1}{m}-\frac{1}{n}}\times Q$$

$$+\overline{\frac{1}{mt}+\frac{1}{nt}+\frac{1}{nm}-\frac{1}{ntr^n}+\frac{2}{mtr^n}-\frac{1}{nmr^n}}\times R-\frac{1-p}{nmt}S;$$

Therefore the difference of thoſe two expreſſions, *viz.*

$$\overline{\frac{1}{mr^m}-\frac{1}{tr^m}-\frac{1}{nr^n}-\frac{n}{mtr^m}+\frac{1}{tr^n}+\frac{1}{mr^n}+\frac{1}{n}}\times Q$$

$$+\overline{\frac{1}{ntr^n}+\frac{1}{nmr^m}-\frac{1}{mtr^m}-\frac{2}{mtr^n}-\frac{1}{nt}-\frac{1}{nm}}+R$$

$+\frac{\overline{1-p}}{nmt}\times S$ will be the value of the reverſion; which may be wrote as follows:

$$\overline{mt-\overline{t-n}\times\overline{m-n}\times\frac{1}{r^n}+\overline{t-m}\times\frac{n}{r^m}}\times\frac{Q}{nmt}-$$

$$\overline{m+t-\overline{m+t-2n}\times\frac{1}{r^n}+n\times\frac{1}{r^m}}\times\frac{R}{nmt}+\overline{\frac{1-p}{nmt}}\times S$$

SCHOLIUM.

If the approximations to the values of the joint lives be applied to the ſolution of this queſtion; then, in this caſe,

The value of the expectants joint lives will (by ſchol. to queſt. 64) be $M\times\overline{1-\frac{m+1}{2t}}+\frac{\overline{m+1}\times\overline{m-1}}{12tr}$ and the value of the three joint lives will be, by ſchol. queſt. 69.

$$N\times\overline{1-\frac{n+1}{2m}}\times\overline{1-\frac{n+1}{2t}}+\frac{n+1}{2m}\times\frac{n-1}{6r}\times\frac{3m+2t-n-3}{2t};$$

which expreſſion, if ſubtracted from the former, will leave a remainder, equally difficult to be applied to numbers, as the two ſeparate expreſſions are; which ſubtraction is therefore omitted.

Now by examp. 1. ſchol. queſt. 64, the two joint lives are worth . 8,364.
And by ſchol. queſt. 69, the three joint lives are worth 5,199

Therefore the reverſion will be . . . 3,165.

CASE 2. When the expectants are of equal ages, and the poſſeſſor elder than either of them;

The ſolution of this may be deduced from that of caſe 1, by making $t=m$ as follows.

$$\overline{mm-\overline{m-n}^{2}\times\frac{1}{r^{n}}}\times\frac{Q}{nmm}-\overline{2m-\overline{m-n}\times\frac{2}{r^{n}}+\frac{n}{r^{m}}}\times\frac{R}{nmm}$$

$$\left(+\frac{1-p}{nmm}\times S\right.$$

The approximations, to the values of the joint lives, are as follow;

The two equal joint lives *per* ſchol. to queſt. 65, will be $M\times\frac{m-1}{2m}+\frac{\overline{m+1}\times\overline{m-1}}{12mr}$ and the value of the three joint lives, the two younger of which are equal, will be

$$N\times\overline{1-\frac{n+1}{2m}}\times\overline{1-\frac{n+1}{2m}}+\frac{n-1}{6r}\times\frac{n+1}{2m}\times\frac{4m-n-3}{2m}$$

by queſt. 70. But in this caſe alſo the ſubtracting the latter expreſſion from the former will not much ſhorten the numerical proceſs.

CASE 3. When the poſſeſſor is younger than either of the expectants, their ages being unequal.

Here we muſt call the complement of the poſſeſſor's life t, and thoſe of the expectants m and n.

Then the value of the joint lives of the expectants will be

$$P + \overline{\frac{1}{nr^n} - \frac{1}{mr^n} - \frac{1}{n} - \frac{1}{m}} \times Q + \overline{\frac{1}{nm} - \frac{1}{mnr^n}} \times R$$

From which, the value of the three joint lives being taken, will leave

$$\overline{\frac{1}{tr^n} + \frac{1}{t} - \frac{n}{mtr^n}} \times Q +$$

$$\overline{\frac{1}{ntr^n} - \frac{2}{mtr^n} - \frac{1}{mt} - \frac{1}{nt}} \times R + \frac{1-p}{nmt} S$$

the value of the reverſion, which may be reduced to

$$\overline{mn + \overline{m-n} \times n} \times \frac{1}{r^n} \times \frac{Q}{nmt} - \overline{n + m + \overline{2n-m} \times \frac{1}{r^n}} \times \frac{R}{nmt}$$

$$\Big(+ \frac{1-p}{umt} \times S$$

Now if the approximations to the values of the joint lives be applied to this caſe; then the joint lives of the expectants will (*per* ſchol. to queſt. 64) be

$$N \times \overline{1 - \frac{n+1}{2m} + \frac{\overline{n+1} \times \overline{n-1}}{12rm}}$$; and the value of the three joint lives $N \times \overline{1 - \frac{n+1}{2m}} \times \overline{1 - \frac{n+1}{2t}} + \frac{n+1}{2m} \times \frac{n-1}{2r} \times$ $\frac{2m+2t-n-3}{6t}$, which may be wrote

$$N \times \overline{1 - \frac{n+1}{2m}} \times \overline{1 - \frac{n+1}{2t}} + \frac{\overline{n+1} \times \overline{n-1}}{12rm} \times \frac{2m+2t-n-3}{2t};$$

which laſt expreſſion, being ſubtracted from the value of the joint lives of the expectants, will leave

$$N \times \overline{1 - \frac{n+1}{2m}} \times \overline{1 - \overline{1 - \frac{n+1}{2t}}} + \frac{\overline{n+1} \times \overline{n-1}}{12rm} \times$$

$$\Big(\overline{1 - \frac{2m+2t-n-3}{2t}}:$$

But $1-\overline{1-\frac{n+1}{2t}}=\frac{n+1}{2t}$, & $1-\frac{2m+2t-n-3}{2t}=$

$$-\frac{2m-n-3}{2t}$$

whence the value of the reverſion will be

$$N\times\overline{1-\frac{n+1}{2m}}\times\frac{n+1}{2t}-\frac{\overline{n+1}\times\overline{n-1}}{12rm}\times\frac{2m-n-3}{2t},$$

Or $N\times\overline{\overline{1-\frac{n+1}{2m}}-\frac{\overline{n-1}\times\overline{2m-n-3}}{12rm}}\times\frac{n+1}{2t}$: which,

becauſe $1-\frac{n+1}{2m}=\frac{2m-n-1}{2m}$ will become

$$\overline{N\times\frac{2m-n-1}{2m}-\frac{\overline{n-1}\times\overline{2m-n-3}}{12rm}}\times\frac{n+1}{2t}$$

But, ſince $2m-n-3$ differs but 2 from $2m-n-1$, and the diviſor $12rm\times 2t$ is great in compariſon of the difference; let $2m-n-1$ be wrote for $2m-n-3$, and the reverſion will be

$$\left(\overline{N\times\frac{2m-n-1}{2m}-\frac{n-1}{6r}\times\frac{2m-n-1}{2m}}\times\frac{n+1}{2t}=\right)\overline{N-\frac{n-1}{6r}}$$
$$\left(\times\frac{\overline{n+1}\times\overline{2m-n-1}}{4mt}\right.$$

Which, expreſſed in words at length, will ſtand as follows:

The rule to find the value of the reverſion of an annuity, on two joint lives after one; when the expectants are ſeverally elder than the poſſeſſor, and of unequal ages, having the value of the oldeſt life given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which each life wants of 86, be called their complements of life; and let the ſum of one pound, and its intereſt for one year, be called the rate.

From the complement of the elder expectant's life ſubtract one, and divide the remainder by ſix times the rate, or let this quotient be found by table the laſt; let the quotient be taken from the value of the oldeſt life, calling the remainder R.

From twice the complement of the younger expectant's life, take the complement of the elder expectant and one; multiply the remainder by the complement of the elder expectant more one; and that product by the above found remainder R; divide this laſt product by four times the product of the complements of the poſſeſſor and younger expectant, ſo ſhall the quotient be the value of the reverſion required.

EXAMPLE.

If the reverſion of an annuity for two joint lives of 54 and 66, after a life of 43, allowing 4 *l. per Cent.* be required,

Then 7,333 will be the value of the eldeſt life;

(86— 43=) 43 will be the complement of the poſſeſſor's life,

(86— 54=) 32 that of the younger expectant,

(86— 66=) 20 that of the elder expectant,

And (1+,04=) 1,04 the rate.

Then (20— 1=) 19 being divided by (6×1,04=) 6,24 will quote 3,045; which, taken from 7,333, will leave 4,288 for the remainder *R*.

And (2×32—20—1=64—21=) 43 being multiplied by (20+1=) 21, will produce 903; which being alſo multiplied by (*R*=) 4,288 produces 3872,064; which laſt product being divided by (4×32×43=) 5504; will quote 0,7035, the value of the reverſion required.

CASE 4. When the expectants are of equal ages, and the poſſeſſor younger than either of them.

The ſolution of this caſe may be deduced from that of the former, by making $m=n$, as follows:

$$\frac{Q}{t}\overline{-2n+\frac{n}{rn}}\times\frac{R}{nnt}+\frac{1-p}{nnt}S$$

Now the approximation to the value of the two equal joint lives of the expectants will (*per* queſt. 65) be

$N\times\frac{n-1}{2n}+\frac{\overline{n+1}\times n-1}{12rn}$; and the value of the three joint lives $N\times\frac{n-1}{2n}\times\frac{2t-n-1}{2t}+\frac{\overline{n+1}\times\overline{n-1}}{1{\cdot}2rn}\times\frac{2t+n-3}{2t}$, (by queſt. 71).

Whence, by ſubtraction, the value of the reverſion will be

$$N\times\frac{n-1}{2n}\times\overline{1-\frac{2t-n-1}{2t}}+\frac{\overline{n+1}\times\overline{n-1}}{12rn}\times\overline{1-\frac{2t+n-3}{2t}}$$

But $1-\frac{2t-n-1}{2t}=\frac{n+1}{2t}$ and $1-\frac{2t+n-3}{2t}=-\frac{n-3}{2t}$;

Therefore the value of the reverſion will become

$$N\times\frac{n-1}{2n}\times\frac{n+1}{2t}-\frac{\overline{n+1}\times\overline{n-1}}{12rn}\times\frac{n-3}{2t},$$

Or $\overline{N-\frac{n-3}{6r}}\times\frac{\overline{n+1}\times\overline{n-1}}{4nt}$:

Alſo, becauſe $(\overline{n+1}\times\overline{n-1}=)\ nn-1$ differs from nn only by unity, and is to be divided by $4nt$, a number great in reſpect thereto; therefore let nn be wrote for $\overline{n+1}\times\overline{n-1}$, and the value of the reverſion will become,

$\left(\overline{N-\frac{n-3}{6r}}\times\frac{nn}{4nt}=\right)\overline{N-\frac{n-1}{6r}}\times\frac{n}{4t}$ which, in words at length, follows:

The rule to find the value of the reverſion of an annuity, on two equal joint lives after one, when the expectants are elder than the poſſeſſor; having the value of the ſingle life of one of the expectants given, allowing compound intereſt at a given rate; and ſuppoſing the decrements of life to be equal.

Let the number of years, which each life wants of 86, *be called their complements of life; and let the ſum of one pound, and its intereſt for one year, be called the rate.*

From the complement of the expectant's life ſubtract three, and divide the remainder by ſix times the rate, or find the quotient by table the laſt.

Subtract this quotient from the given value of the expectant's life; multiply the remainder by the complement of the expectant's life; and divide the product by four times the complement of the poſſeſſor's life, ſo ſhall the quotient be the value of the reverſion required.

EXAMPLE.

If the lives in reverſion are each 66, the poſſeſſor 43, and intereſt four *per Cent.*

Then 7,333 will be the value of each expectant's life;
(86—43=) 43, the comp. of the poſſeſſor's life,
(86—66=) 20, that of each expectants,
And (1+,04=) 1,04 the rate.
Now (20−3=) 17, being divided by (6×1,04=) 6,24, will quote 2,724.

And (7,333−2,724=) 4,609, being multiplied by 20, produces 92,180; which, being divided by (4×43=) 172, will quote 0,536, the value of the reverſion required.

CASE 5. When the expectants are, one elder; and the other younger, than the poſſeſſor.

Here the complement of the poſſeſſors life muſt be called *m*, and thoſe of the expectants *t* and *n*.

Then the value of the joint lives of the expectants will be

$$P+\overline{\frac{1}{nr^n}-\frac{1}{tr^n}-\frac{1}{n}-\frac{1}{t}}\times\mathcal{Q}+\overline{\frac{1}{nt}-\frac{1}{ntr^n}}\times R$$

From which, taking the value of the three joint lives, there will remain

$$\overline{\frac{1}{mrn}+\frac{1}{m}-\frac{n}{mtr^n}}\times\mathcal{Q}+$$

$$(\overline{\frac{1}{nmr^n}-\frac{2}{mtr^n}-\frac{1}{mt}-\frac{1}{mn}}\times R+\frac{1-p}{nmt}S$$

the value of the reverſion, which may be reduced to

$$\overline{nt+\overline{t-n}\times n\times\frac{1}{r^n}}\times\frac{\mathcal{Q}}{nmt}-\overline{n+t+\overline{2n-t}\times\frac{1}{r^n}}\times\frac{R}{nmt}+\frac{1-p}{nmt}S.$$

Now the approximation, to the value of the joint lives of the expectants, will be

$N\times\overline{1-\frac{n+1}{2t}}+\frac{\overline{n+1}\times\overline{n-1}}{12rt}$ and the approximation to the value of the joint lives,

$$N\times\overline{1-\frac{n+1}{2m}}\times\overline{1-\frac{n+1}{2t}}+\frac{\overline{n+1}\times\overline{n-1}}{12\,t}\times\frac{2m+2t-n-3}{2m};$$

the difference of which will be

N×

$$N\times 1-\frac{\overline{n+1}}{2t}\times\frac{n+1}{2m}-\frac{\overline{n+1}\times\overline{n-1}}{12rt}\times\frac{2t-n-3}{2m}.$$

Or $\overline{N\times 1-\frac{\overline{n+1}}{2t}-\frac{\overline{n-1}\times\overline{2t-n-3}}{12r}}\times\frac{n+1}{2m}.$

Again, (becauſe $1-\frac{n+1}{2t}=\frac{2t-n-1}{2t}$) the above may be wrote as below

$\overline{N\times\frac{2t-n-1}{2t}-\frac{n-1}{6r}\times\frac{2t-n-3}{2t}}\times\frac{n+1}{2m}$, which (writing $2t-n-1$ for $2t-n-3$ as in caſe 3) will become

$\overline{N}-\frac{\overline{n-1}}{6r}\times\frac{\overline{n+1}\times\overline{2t-n-1}}{4mt}$, the value of the reverſion required.

This may be expreſſed by the ſame words as in the rule for caſe 3; to which the reader is referred.

EXAMPLE.

If the expectants are ſeverally 43 and 66 years of age; the poſſeſſor 54; and the rate of intereſt four *per Cent.* Then 7,333 will be the value of the oldeſt life;

(86—54=) 32 will be the complement of the poſſeſſor's life,

(86—43=) 43 that of the younger expectant

(89—66=) 20 that of the elder expectant,

And (1+,04=) 1,04 the rate.

Then (20—1=) 19, being divided by (6×1,04=) 6,24, will quote 3,045; which, taken from 7,333, will leave 4,288, for the remainder *R*.

And (2×43–20–1=86–21=) 65, being multiplied by (20+1=) 21, will produce 1365; which, being alſo multiplied by the remainder (*R*) 4,288, produces 5853,120;

which laſt product, being divided by (4×32×43=)5504, will quote 1,063, the value of the reverſion required.

CASE 6. When the elder expectant and the poſſeſſor are of the ſame age.

This caſe may be ſolved from the laſt, by writing n for m; and the value of the reverſion will be

$$\overline{nt+\overline{t-n}}\times\frac{n}{r}\times\frac{Q}{nnt}-\overline{n+t+\overline{2n-t}}\times\frac{1}{r^n}\times\frac{R}{nnt}+\frac{1-p}{nnt}S;$$

And the approximation thereto $N-\frac{\overline{n-1}}{6r}\times\frac{\overline{n+1}\times\overline{2t-n-1}}{4nt}$ which may be expreſſed by the ſame words, as the rule in caſe 3.

CASE 7. When the younger expectant and the poſſeſſor are of the ſame age.

This may be alſo ſolved from caſe 5, by writing m for t; and the value of the reverſion will be

$$\overline{nm+\overline{m-n}}\times\frac{n}{r^n}\times\frac{Q}{nmm}-\overline{n+m+\frac{2n-m}{r^n}}\times\frac{R}{nmm}+\frac{1-p}{nmm}S$$

And the approximation thereto $N-\frac{\overline{n-1}}{6r}\times\frac{\overline{n+1}\times\overline{2m-n-1}}{4mm}$

This is alſo expreſſible by the words given in the rule to caſe 3.

CASE 8. When the expectants and poſſeſſor are of equal ages.

Then calling the complement of their lives n, the ſolution may be obtained from any of the foregoing caſes (ſuppoſe the firſt) by writing n for t and m; alſo $\frac{1}{r^n}$ for $\frac{1}{r^m}$; as follows,

$$\frac{Q}{n}-\frac{2+p}{nn}R+\frac{1-p}{nnn}S.$$

Now the approximation to the value of this reverſion may be deduced from that in caſe 4, by writing *n* for *t*, *viz.*

$$N-\overline{\frac{n-3}{6r}}\times\frac{n}{4n}, \text{ Or } N-\overline{\frac{n-3}{6r}}\times\frac{1}{4}$$ which, in words at length, follows:

The rule to find the value of the reverſion of an annuity, on two equal joint lives after another life of the ſame age; having the value of a ſingle life of that age given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, *be called the complement of life; and let the ſum of one pound, and its intereſt for one year, be called the rate.*

From the complement of life ſubtract three, and divide the remainder by ſix times the rate, or find this quotient by the laſt table.

Subtract this quotient from the given value of the ſingle life; then ſhall one fourth part of the remainder be the value of the reverſion required.

EXAMPLE.

If the given age be 66, and intereſt four *per Cent.*

Then 7,333 will be the value of the ſingle life,

(86—66=) 20 will be the complement of life,

And (1+,04=) 1,04 the rate.

If (20—3=) 17, be divided by (6×1,04=) 6,24, the quotient will be 2,724.

And 7,333—2,724=4,609, the fourth part of which, *viz.* 1,152, will be the value of the annuity required.

QUESTION XCVI.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the joint lives of three perſons of equal ages, after the deceaſe of a fourth perſon of the ſame age?

SOLUTION.

Let the complement of their lives be denoted by n;

Then the probability of the poſſeſſor's dying in the firſt, ſecond, third, &c. year will be

$$1-\frac{n-1}{n},\ 1-\frac{n-2}{n},\ 1-\frac{n-3}{n}\ \&c.$$

And the probability of the joint continuance of the three expectants lives, will, for the firſt, ſecond, third, &c. year be

$$\frac{\overline{n-1}^{\,3}}{n^3},\ \frac{\overline{n-2}^{\,3}}{n^3},\ \frac{\overline{n-3}^{\,3}}{n^3}\ \&c.$$

whence the values of the firſt, ſecond, third, &c. payment of the reverſion will be

$$\frac{\overline{n-1}^{\,3}}{n^3}-\frac{\overline{n-1}^{\,4}}{n^4},\ \frac{\overline{n-2}^{\,3}}{n^3}-\frac{\overline{n-2}^{\,4}}{n^4},\ \frac{\overline{n-3}^{\,3}}{n^3}-\frac{\overline{n-3}^{\,4}}{n^4}\ \&c.$$

And therefore, the value of the required reverſion may be found by ſubtracting the value of the four equal joint lives, from the value of the three equal joint lives.

Now retaining the ſymbols uſed in corol. to queſt. 75.

$$N^{\mathrm{iii}}=P-\frac{3}{n}Q+\frac{3}{nn}R-\frac{1-p}{n^3}S,$$

$$N^{\mathrm{iv}}=P-\frac{4}{n}Q+\frac{6}{nn}R-\frac{4}{n^3}S+\frac{1-p}{n^4}T;$$

And therefore the value of the reverſion required will be

$$\frac{1}{n}Q-\frac{3}{nn}R+\frac{3+p}{n^3}S-\frac{1-p}{n^4}T.$$

COROL.

Retaining the ſame ſymbols, and writing $\mathfrak{R}$ for the reverſion of one life after one; $\mathfrak{R}^{ii}$ for the reverſion of two equal joint lives after one; $\mathfrak{R}^{iii}$ for the reverſion of three equal joint lives after one, &c.

Since $\mathfrak{R}=\frac{1+p}{n}Q-\frac{1-p}{nn}R$, By queſt. 94

$$\mathfrak{R}^{ii}=\frac{1}{n}Q-\frac{2+p}{nn}R+\frac{1-p}{n^3}S, \quad 95$$

$$\mathfrak{R}^{iii}=\frac{1}{n}Q-\frac{3}{nn}R+\frac{3+p}{n^3}S-\frac{1-p}{n^4}T; \quad 96$$

$$\text{Therefore}\left\{\begin{array}{l}\mathfrak{R}^{iv}=\frac{1}{n}Q-\frac{4}{nn}R+\frac{6}{n^3}S-\frac{4+p}{n^4}T+\frac{1-p}{n^5}V;\\ \mathfrak{R}^{v}=\frac{1}{n}Q-\frac{5}{nn}R+\frac{10}{n^3}S-\frac{10}{n^4}T+\frac{5+p}{n^5}V-\\ \qquad\qquad\qquad\qquad\qquad\qquad\left(\frac{1-p}{n^6}W;\right.\end{array}\right.$$

And the value of the reverſion of m equal joint lives, after the failure of one life of the ſame age, will be

$$\frac{1}{n}Q-\frac{m}{n^2}R+\frac{m\cdot\overline{m-1}}{n^2\cdot 2n}S-\frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{n^3\cdot 2n\cdot 3n}T\ (m-1).$$

$$\left(\pm\frac{m+p}{n^m}Y\mp\frac{1-p}{n^{m+1}}Z.\right.$$

In which expreſſion Y and Z denote the $\overline{m+1}$th and $\overline{m+2}$th factors in the ſeries above quoted P, Q, R, &c. and the ſigns throughout the whole will be alternately $+$ and $-$.

QUESTION XCVII.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, on the the longeſt of two lives of given ages, after the deceaſe of the preſent poſſeſſor, whoſe age is alſo given.

SOLUTION.

Let the complement of the life of the preſent poſſeſſor be denoted by n; and thoſe of the expectants by t and m.

Then the probability of the poſſeſſor's dying in the firſt, ſecond, third, &c. years, will be

$$1-\frac{n-1}{n},\ 1-\frac{n-2}{n},\ 1-\frac{n-3}{n}\ \&c.$$

And by queſt. 76, the probability of one (at leaſt) of the expectants ſurviving the firſt, ſecond, third, &c. years will be

$$\frac{m-1}{m}+\frac{t-1}{t}-\frac{\overline{m-1}\times\overline{t-1}}{mt},$$

$$\frac{m-2}{m}+\frac{t-2}{t}-\frac{\overline{m-2}\times\overline{t-2}}{mt},$$

$$\frac{m-3}{m}+\frac{t-3}{t}-\frac{\overline{m-3}\times\overline{t-3}}{mt}\ \&c.$$

which probabilities, being ſeverally multiplied into the correſponding probabilities of the poſſeſſor's dying, will give the values of the firſt, ſecond, third, &c. payments of the reverſion, *viz.*

$$\frac{m-1}{m}+\frac{t-1}{t}-\frac{\overline{m-1}\times\overline{t-1}}{mt}-\frac{\overline{n-1}\times\overline{m-1}}{nm}-\frac{\overline{n-1}\times\overline{t-1}}{nt}$$

$$\frac{m-2}{m}+\frac{t-2}{t}-\frac{\overline{m-2}\times\overline{t-2}}{mt}-\frac{\overline{n-2}\times\overline{m-2}}{nm}-\frac{\overline{n-2}\times\overline{t-2}}{nt}$$

$$\frac{m-3}{m}+\frac{t-2}{t}-\frac{\overline{m-3}\times\overline{t-3}}{mt}-\frac{\overline{n-3}\times\overline{m-3}}{nm}-\frac{\overline{n-3}\times\overline{t-3}}{nt}$$

&c. &c. &c. &c. &c.

$$\left(+\frac{\overline{n-1}\times\overline{m-1}\times\overline{t-1}}{nmt},\right.$$

$$\left(+\frac{\overline{n-2}\times\overline{m-2}\times\overline{t-2}}{nmt},\right.$$

$$\left(+\frac{\overline{n-3}\times\overline{m-3}\times\overline{t-3}}{nmt};\right.$$

&c.

The preſent value of which expreſſion, being compared with the firſt expreſſion (given in queſt. 80) for the value of the longeſt of three lives, will appear to differ therefrom, by

$\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}$ &c. the value of the life of the poſſeſſor.

Therefore; if, from the value of the longeſt of the three lives of the poſſeſſor and two expectants, be taken the value of the poſſeſſor's life, the remainder will be the value of the reverſion, of the longeſt of two lives after one life.

The application of which rule to the foregoing ſolutions will admit of eight caſes, as in queſt. 95.

CASE I. When the poſſeſſor is older than either of the expectants, their ages being unequal.

By the ſolution of queſt. 80, the value of the longeſt of three lives is

P+

$$P+\overline{\frac{1}{tr^t}+\frac{1}{tr^m}+\frac{n}{mtr^n}}\times\mathcal{Q}+\overline{\frac{2}{mtr^n}+\frac{1}{mtr^m}}\times R-\frac{1-p}{nmt}S$$

And (by queſt. 56) the value of the poſſeſſor's life will be $P+\overline{\frac{1}{nr^n}-\frac{1}{n}}\times\mathcal{Q}$; which, ſubtracted from the former, leaves

$$\overline{\frac{1}{tr^t}+\frac{1}{tr^m}+\frac{n}{mtr^n}+\frac{1}{n}-\frac{1}{nr^n}}\times\mathcal{Q}$$

$\left(+\overline{\frac{2}{mtr^n}+\frac{1}{mtr^m}}\times R-\frac{1-p}{nmt}S\right.$, the value of the reverſion, which may be deduced to

$$\overline{\overline{\frac{1}{r^t}+\frac{1}{r}}\times nm+mt-\overline{mt}-nn}\times\frac{1}{rn}\times\frac{\mathcal{Q}}{nmt}+\overline{\frac{2}{r^n}+\frac{1}{r^m}}$$

$$\left(\times\frac{nR}{nmt}-\frac{1-p}{nmt}S\right.$$

Nothing would be ſaved (in the numerical proceſs) by ſubtracting the ſymbol of the poſſeſſor's life from the approximation to the value of the longeſt of the three lives, found in queſt. 81.

Therefore from the approximate value of the longeſt of the three lives - - - - -	15,141,
Take the value of the poſſeſſor's life	7,333
The remainder will be the value of the reverſion	7,808.

It remains to compare theſe with the true reſult, *viz.*

From the true value of the longeſt of the three lives found in queſt. 80 - - - - -	15,190
Take the value of the poſſeſſor's life	7,333
Remains the true value of the reverſion	7,857

It appears from the ſolution of the above caſe, that, where the value of the poſſeſſor's life is given (as in the tables annexed) this numerical proceſs will be ſoon performed;

formed; and therefore the next 6 caſes are here omitted, and we ſhall proceed to the 8th.

CASE 8. When the poſſeſſor and expectants are of equa ages.

This may be ſolved from caſe 1, by making $\frac{1}{r^t} = \frac{1}{r^m} = \frac{1}{r^n} = p$; and $m = t = n$; thus

$\frac{1+2p}{n} Q + \frac{3p}{nn} R - \frac{1-p}{n^3} S.$

Alſo, if from the approximation to the value of the longeſt of three equal lives, *viz.* $N + \frac{\overline{3n+1} \times \overline{n+1}}{4nn} \times \overline{N - \frac{n-1}{6r}}$; the value of the poſſeſſor's life (N) be taken, the remainder $\frac{\overline{3n+1} \times \overline{n+1}}{4nn} \times \overline{N - \frac{n-1}{6r}}$ will be the value of the reverſion.

The giving of a rule, in words, for which, will be unneceſſary; becauſe it will differ from that in queſt. 84 and 85, only, in not adding the value of the ſingle life, to the quotient, found as therein directed.

QUESTION XCVIII.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, on the longeſt of three equal lives, after the deceaſe of a fourth perſon of the ſame age.

SOLU-

SOLUTION.

Let the complement of their lives be denoted by n;

Then the probabilities of the poſſeſſor's dying in the firſt, ſecond, third, &c. year, will be

$$1-\frac{n-1}{n},\ 1-\frac{n-2}{n},\ 1-\frac{n-3}{n}\ \&c.$$

And by arguing, as in queſt. 86, the probabilities of one, at leaſt, of the expectants ſurviving the firſt, ſecond, third, &c. year, will be

$$3\times\frac{n-1}{n}-3\times\frac{\overline{n-1}^2}{nn}+\frac{\overline{n-1}^3}{n^3},$$

$$3\times\frac{n-2}{n}-3\times\frac{\overline{n-2}^2}{nn}+\frac{\overline{n-2}^3}{n^3},$$

$$3\times\frac{n-3}{n}-3\times\frac{\overline{n-3}^2}{nn}+\frac{\overline{n-3}^3}{n^3}\ \&c.$$

which probabilities, being ſeverally multiplied into the correſponding probabilities of the poſſeſſor's dying, will give the firſt, ſecond, third, &c. payment.

$$3\times\frac{n-1}{n}-6\times\frac{\overline{n-1}^2}{nn}+4\times\frac{\overline{n-1}^3}{n^3}-\frac{\overline{n-1}^4}{n^4},$$

$$3\times\frac{n-2}{n}-6\times\frac{\overline{n-2}^2}{nn}+4\times\frac{\overline{n-2}^3}{n^3}-\frac{\overline{n-2}^4}{n^4},$$

$$3\times\frac{n-3}{n}-6\times\frac{\overline{n-3}^2}{nn}+4\times\frac{\overline{n-3}^3}{n^3}-\frac{\overline{n-3}^4}{n^4}\ \&c.$$

which differ from the like expreſſions found for the value of the longeſt of four equal lives by the value of the ſingle life $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}$ &c.

And therefore the value of the required reverſion will be found by ſubtracting the value of the ſingle life from the value of the longeſt of the four equal lives.

Now, if we uſe the ſame ſymbols as in the corol. to queſt. 86,

$$L\cdot N^{\mathrm{iv}} = P + \frac{4p}{n}Q + \frac{6p}{nn}R + \frac{4p}{n^3}S - \frac{1-p}{n^4}T,$$

$$N = P - \frac{1-p}{n}Q;$$

Therefore the value of the reverſion of an annuity, for the longeſt of three equal lives, after one, will be

$$\frac{1+3p}{n}Q + \frac{6p}{nn}R + \frac{4p}{n^3}S - \frac{1-p}{n^4}T.$$

COROL.

If $L\cdot\mathfrak{R}^{\mathrm{ii}}$ denote the reverſion of the longeſt of two equal lives after one; $L\cdot\mathfrak{R}^{\mathrm{iii}}$ the reverſion of the longeſt of three equal lives after one, &c. then

ſince $\mathfrak{R} = \frac{1+p}{n}Q - \frac{1-p}{nn}R,$ queſt. 94

$L\cdot\mathfrak{R}^{\mathrm{ii}} = \frac{1+2p}{n}Q + \frac{3p}{nn}R - \frac{1-p}{n^3}S,$ 97

$L\cdot\mathfrak{R}^{\mathrm{iii}} = \frac{1+3p}{n}Q + \frac{6p}{nn}R + \frac{4p}{n^3}S - \frac{1-p}{n^4}T;$

then $L\cdot\mathfrak{R}^{\mathrm{iv}} = \frac{1+4p}{n}Q + \frac{10p}{nn}R + \frac{10p}{n^3}S + \frac{5p}{n^4}T -$

$\left(\frac{1-p}{n^5}V\right.$:

And the value of the reverſion of the longeſt of m equal lives, after one life of equal age with the former, will be

$\frac{1+mp}{n}Q + \frac{\overline{m+1}\cdot m}{n\cdot 2n}R + \frac{\overline{m+1}\cdot m\cdot\overline{m-1}}{n\cdot 2n\cdot 3n}S$ (m)

$\left(-\frac{1-p}{n^{m+1}}Z\right.$

Where Z is the $\overline{m+2}$th factor of the ſeries *P*, *Q*, *R*, &c. mentioned in queſt. 20; and the term into which it is to be multiplied, will always be negative.

QUESTION XCIX.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the life of a perſon of a given age, after the joint lives of two perſons, whoſe ages are alſo given.

SOLUTION.

Let the complements of the two lives in poſſeſſion be *m* and *n*; and that of the expectant *t*.

Now ſince the probability of both of the poſſeſſors living, to the end of the firſt, ſecond, third, &c year, is

$$\frac{\overline{m-1}\times\overline{n-1}}{mn},\ \frac{\overline{m-2}\times\overline{n-2}}{mn},\ \frac{\overline{m-3}\times\overline{n-3}}{mn}\ \&c.$$

Therefore the probability, that one of them, at leaſt, will fail, in the firſt, ſecond, third, &c. year, will be

$$1-\frac{\overline{m-1}\times\overline{n-1}}{mn},\ 1-\frac{\overline{m-2}\times\overline{n-2}}{mn};\ 1-\frac{\overline{m-3}\times\overline{n-3}}{mn}\ \&c$$

And the probability of the continuance of the expectant's life, for one, two, three, &c. years, will be

$$\frac{t-1}{t},\ \frac{t-2}{t},\ \frac{t-3}{t}\ \&c.$$

which probabilities, being ſeverally multiplied by the correſponding probabilities of one of the poſſeſſors failing, will give the value of the firſt, ſecond, &c. payments, of the reverſion, *viz.*

$$\frac{t-1}{t} - \frac{\overline{m-1}\times\overline{n-1}\times\overline{t-1}}{mnt},\ \frac{t-2}{t} - \frac{\overline{m-2}\times\overline{n-2}\times\overline{t-2}}{mnt}, \&c.$$

the preſent value, of which payments, will be

$$\frac{t-1}{tr} - \frac{\overline{m-1}\times\overline{n-1}\times\overline{t-1}}{mntr} + \frac{t-2}{tr^2} - \frac{\overline{m-2}\times\overline{n-2}\times\overline{t-2}}{mntr^2} \&c.$$

where $\frac{t-1}{tr} + \frac{t-2}{tr^2} + \frac{t-3}{tr^3}$ &c. is the value of the expectants life,

and $\frac{\overline{m-1}\times\overline{n-1}\times\overline{t-1}}{mntr} + \frac{\overline{m-2}\times\overline{n-2}\times\overline{t-2}}{mntr^2}$ &c. is the value of the three joint lives, of the two poſſeſſors and the expectant.

Therefore; if, from the value of the expectant's life, the value of the three joint lives (of the two poſſeſſors and the expectant) be taken; the remainder will be the value of one life, after two joint lives.

Now here (as in queſt. 97) it will appear, that, when a table of the values of ſingle lives is at hand, nothing will be ſaved, in the numerical ſolution, by finding the difference of thoſe two values, as expreſſed in ſymbols: it is therefore here thought ſufficient to give an example, and anſwer it by the former reſults.

What is the value of the reverſion of an annuity, to continue during the life of a perſon aged 43 years, after the joint lives of two perſons of the reſpective ages of 54 and 66; allowing compound intereſt at four *per Cent?*

Firſt, by the true method,

From the value of the expectant's life (queſt. 56)	12,683,
Take the value of the three joint lives (queſt. 69)	5,152,
Remains the value of the reverſion	7,531.

Secondly, by the approximation,

From the value of the expectant's life	12,683,
Take the value of the three joint lives	5,199,
Remains the value of the reverſion	7,484.

COROL.

If the given lives are equal, and the ſame ſymbols be uſed as in the corol. to queſt 75 ; then

$$\dot{N} = P - \frac{1-p}{n} Q,$$

$$N^{\text{iii}} = P - \frac{3}{n} Q + \frac{3}{nn} R - \frac{1-p}{n^3} S;$$

therefore the value of the required reverſion will be

$$\frac{2+p}{n} Q - \frac{3}{nn} R + \frac{1-p}{n^3} S.$$

QUESTION C.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the life of a perſon of a given age, after the joint lives of three other perſons, each of the ſame age.

SOLUTION.

Let the complement of thoſe lives be n.

Then the probability that one, at leaſt, of the three poſſeſſors will die in the firſt, ſecond, third, &c. year, will be

$$1 - \frac{\overline{n-1}^3}{n^3},\ 1 - \frac{\overline{n-2}^3}{n^3},\ 1 - \frac{\overline{n-3}^3}{n^3}\ \text{\&c.}$$

which, being ſeverally multiplied by the reſpective probabilities of the expectant's living to the end of the firſt, ſecond, third, &c. year, will give the firſt, ſecond, third, &c. payment of the reverſion, *viz.*

$$\frac{n-1}{n}-\frac{\overline{n-1}^4}{n^4},\ \frac{n-2}{n}-\frac{\overline{n-2}^4}{n^4};\ \frac{n-3}{n}-\frac{\overline{n-3}^4}{n^4}.$$

Therefore; if, from the value of the ſingle life, the value of the four joint lives be taken, the remainder will be the value of the reverſion.

Now, if the ſame ſymbols be uſed as in corol. to queſt. 75, then

$$N=P-\frac{1-p}{n}Q;$$

$$N^{\text{iv}}=P-\frac{4}{n}Q-\frac{6}{nn}R-\frac{4}{n^3}S+\frac{1-p}{n^4}T;$$

therefore $\frac{3+p}{n}Q-\frac{6}{nn}R+\frac{4}{n^3}S-\frac{1-p}{n^4}T$ will be the value of the reverſion required.

COROL.

If ${}^{\text{ii}}\mathfrak{R}$ denote the value of the reverſion of one life, after two equal joint lives; the ${}^{\text{iii}}\mathfrak{R}$ the reverſion of one life, after three ſuch lives &c. then

ſince $\mathfrak{R}=\frac{1+p}{n}Q-\frac{1-p}{nn}R,$ queſt. 94

${}^{\text{ii}}\mathfrak{R}=\frac{2+p}{n}Q-\frac{3}{nn}R+\frac{1-p}{n^3}S,$ 99

$${}^{\text{iii}}\mathfrak{R}=\frac{3+p}{n}Q-\frac{6}{nn}R+\frac{4}{n^3}S-\frac{1-p}{n^4}T;$$

$${}^{\text{iv}}\mathfrak{R}=\frac{4+p}{n}Q-\frac{10}{nn}R-\frac{10}{n^3}S-\frac{5}{n^4}T+\frac{1-p}{n^5}V.$$

And the value of the reverſion of one life, after m joint lives of equal ages, will be

$$\frac{m+p}{n}Q-\frac{\overline{m+1}\cdot m}{n\cdot 2n}R+\frac{\overline{m+1}\cdot m\cdot\overline{m-1}}{n\cdot 2n\cdot 3n}S\,(m)\pm\frac{1-p}{n^{\overline{m+1}}}Z,$$

where Z is the $\overline{m+2}$th factor of the ſeries P, Q, R, &c. whoſe values are given in queſt. 20, and the terms will be alternately + & –, throughout the whole expreſſion.

QUESTION CI.

Suppoſing the the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the life of a perſon of a given age, after the longeſt liver of two perſons, whoſe ages are alſo given.

SOLUTION.

Let the complements of the lives in poſſeſſion be m and n, and that of the expectant t.

Now the probabilities of both of the poſſeſſors dying in the firſt, ſecond, third, &c. years, will (*per* corol. queſt. 29) be

$$\left(\overline{1-\frac{n-1}{n}}\times\overline{1-\frac{m-1}{m}}=\right)1-\frac{n-1}{n}-\frac{m-1}{m}+\frac{\overline{n-1}\times\overline{m-1}}{nm},$$

$$\left(\overline{1-\frac{n-2}{n}}\times\overline{1-\frac{m-2}{m}}=\right)1-\frac{n-2}{n}-\frac{m-2}{m}+\frac{\overline{n-2}\times\overline{m-2}}{nm},$$

$$\left(\overline{1-\frac{n-3}{n}}\times\overline{1-\frac{m-3}{m}}=\right)1-\frac{n-3}{n}-\frac{m-3}{m}+\frac{\overline{n-3}\times\overline{m-3}}{nm};$$

&c. &c.

and the probability of the expectant's living to the end of the firſt, ſecond, third, &c. years, is $\frac{t-1}{t}$, $\frac{t-2}{t}$, $\frac{t-3}{t}$ &c. which probabilities, being ſeverally multiplied into the correſponding probabilities of the two poſſeſſors dying, and the preſent values of the products being taken, will give the value of the reverſion, viz.

$$\begin{aligned}&\frac{t-1}{tr}-\frac{\overline{n-1}\times\overline{t-1}}{ntr}-\frac{\overline{m-1}\times\overline{t-1}}{mtr}+\frac{\overline{n-1}\times\overline{m-1}\times\overline{t-1}}{nmtr},\\&+\frac{t-3}{tr^2}-\frac{\overline{n-2}\times\overline{t-2}}{ntr^2}-\frac{\overline{m-2}\times\overline{t-2}}{mtr^2}+\frac{\overline{n-2}\times\overline{m-2}\times\overline{t-2}}{nmtr^2},\\&+\frac{t-3}{tr^3}-\frac{\overline{n-3}\times\overline{t-3}}{ntr^3}-\frac{\overline{m-3}\times\overline{t-3}}{mtr^3}+\frac{\overline{n-3}\times\overline{m-3}\times\overline{t-3}}{nmtr^3};\\&\quad\&c.\qquad\&c.\qquad\&c.\qquad\&c.\end{aligned}$$

Now if the two firſt expreſſions, given in queſtions 76 and 80 for the reſpective values of the longeſt of two and three lives, be compared with this; it will appear to be their difference: and, therefore

If, from the value of an annuity on the longeſt of the three lives of the two poſſeſſors and expectant, the value of an annuity on the longeſt of the two poſſeſſors lives be taken; the remainder will be the value of the reverſion, of one life, after the longeſt of two lives.

The application of this, to the former ſolutions, will likewiſe admit of eight caſes.

CASE 1. When the poſſeſſors are of unequal ages, and both elder than the expectant.

Then, the value of the annuity on the longeſt of the three lives, will (*per* queſt. 80) be

$$P+\overline{\frac{1}{tr^t}+\frac{1}{tr^m}+\frac{n}{mtr^n}}\times Q+\overline{\frac{2}{mtr^n}+\frac{1}{mtr^m}}\times R-\frac{1-p}{nmt}S$$

and the value of the annuity, on the longeſt of the two poſſeſſors lives, will be

$$P+\overline{\frac{1}{mr^m}+\frac{1}{mr^n}}\times Q-\overline{\frac{1}{nm}-\frac{1}{nmr^n}}\times R$$

the difference of which two expreſſions will be

$$\frac{1}{r^t}-\overline{\overline{\frac{1}{m}-\frac{1}{t}}\times\frac{1}{r^m}-\overline{\frac{1}{m}-\frac{n}{mt}}\times\frac{1}{r^n}}\times Q+$$

$$\overline{\frac{1}{nm}+\frac{1}{mtr^m}+\overline{\frac{2}{mt}-\frac{1}{nm}}\times\frac{1}{r^n}}\times R-\frac{1-p}{nmt}S;$$

$$\text{Or }\overline{\frac{nm}{r^t}-\frac{\overline{t-m}\times n}{r^m}-\frac{\overline{t-n}\times n}{r^n}}\times\frac{Q}{nmt}+$$

$$\overline{1+\frac{n}{r^m}+\frac{\overline{2n-t}}{r^n}}\times\frac{R}{nmt}-\frac{\overline{1-p}}{nmt}S;$$

the value of the reverſion required.

EXAMPLE.

A, who is 43 years of age, is entitled to the reverſion of an eſtate of one pound *per annum* for his life, after the deceaſe of his father (who is 66 years of age) and of his mother-in-law (aged 54) who is jointured thereon; I demand the value of *A*'s intereſt in the eſtate, allowing four *per Cent*.

By the firſt method,

From the value of the longeſt of the three given lives, found *per* queſt. 80 - - - -	15,190,
Subtract the value of the longeſt of the two poſſeſſors lives, found *per* queſt. 76 -	11,904:
Remains the value of the reverſion required	3,286.

By the method laſt given ;-

Here $Q=650$; $R=33150$; $S=2535650$; by table queſt. 20. $\frac{1}{r^t}=0,185168$; $\frac{1}{r^m}=0,285056$; $\frac{1}{r^n}=0,456387$ by table of the the preſent worths of one pound $n=20$; $m=32$; $t=43$; $nm=640$; $t-m=11$, $\overline{t-m}\times n=220$; $t-n=23$; $\overline{t-n}\times n=460$; $640\times 0,185168=118,5075$; $220\times 0,285058=61,7128$; $460\times 0,456387=209,9380$; $20\times 0,285058=5,7612$; $2n-t=-3$,; $-3\times 0,456387=-1,3692$:

$$\begin{array}{ll} +118,5075 & +43 \\ -\ 62,7128 & +\ 5,7612 \\ -209,9380 & -\ 1,3692 \\ \hline \end{array}$$

$-154,1433\times 650=-100193$; $+\ 47,3320\times 33650=1569056$; $(1-0,456387=)\ 0,543613\times 2535650=1378412$

$$\begin{array}{ll} \text{Then} & -\ 100193 \\ & +1569056 \\ & -1378412 \\ \hline & +\ 90451, \end{array}$$

being divided by $(20\times 32\times 43=)$ 27520, will quote 3,2867. the value of the reverſion required.

The approximation to the value of the longeſt of the three lives is, *per* queſt. 81,

$$F+\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^2}{4mt}+\overline{M-\frac{m-1}{6r}}\times\frac{m+1}{2t};$$

and that to the value of the longeſt of the two poſſeſſors lives is, *per* queſt. 68, $M+\overline{N-\frac{n-1}{6r}}\times\frac{n+1}{2m}$; therefore the value of the reverſion will be

$F-$

$$F - M + \overline{N - \frac{n-1}{6r}} \times \overline{\frac{\overline{n+1}^{2}}{4mt} - \frac{n+1}{2m}} + \overline{M - \frac{m-1}{6r}} \times \frac{m+1}{2t}:$$

But $\frac{\overline{n+1}^{2}}{4mt} = \frac{n+1}{2m} \times \frac{n+1}{2t}$; and $\left(\frac{n+1}{2t} - 1 =\right) - \frac{2t-n-1}{2t}$;

Then $\overline{N - \frac{n-1}{6r}} \times \overline{\frac{\overline{n+1}^{2}}{4mt} - \frac{n+1}{2m}} = -\overline{N - \frac{n-1}{6r}} \times \frac{2t-n-1}{2t} \times \frac{n+1}{2m}$

and therefore the value of the reverſion will become,

$$F + \overline{M - \frac{m-1}{6r}} \times \frac{m+1}{2t} - M - \overline{N - \frac{m-1}{6r}} \times \frac{2t-n-1}{2t} \times \frac{n+1}{2m}$$

Or $F - M + \frac{1}{2t} \times \overline{\overline{m+1} \times \overline{M - \frac{m-1}{6r}} - \overline{m+1} \times \frac{2t-n-1}{2m} \times \overline{N - \frac{n-1}{6r}}}$

The above rule, expreſſed in words at length, follows;

The rule for finding the preſent value of the reverſion of an annuity, for the life of a perſon of a given age, after the deceaſe of two other perſons of unequal given ages, both elder than the former; having the values of their ſingle lives given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which each perſon wants of 86, *be called their complements of life; and let one pound, and its intereſt for one year, be called the rate.*

From the complement of the younger poſſeſſor's life, ſubtract one; and divide the remainder by ſix times the rate, or find this quotient in the laſt table.

Subtract the quotient from the value of the younger poſſeſſor's life; and multiply the remainder by his complement more one, *reſerving the product.*

From the complement of the elder poſſeſſor's life, ſubtract one; and divide the remainder by ſix times the rate, or find this quotient in table the laſt; ſubtract the quotient from the value of the elder poſſeſſor's life, reſerving the remainder.

From twice the expectant's complement, ſubtract the elder poſſeſſor's complement, and one; multiply the remainder by the elder poſſeſſor's complement more one; *and this product by the remainder above reſerved; divide this laſt product, by twice the younger poſſeſſor's complement.*

From the product, above reſerved, ſubtract the laſt found quotient, and divide the remainder by twice the expectant's complement.

To the difference of the values of the ſingle lives, of the expectant and younger poſſeſſor, add the quotient laſt found; and their ſum will be the value of the reverſion required.

EXAMPLE.

If the poſſeſſor's be ſeverally of the ages of 54 and 66 years, the expectant 43, and intereſt four *per Cent.* as above.

Then 12,683 will be the value of the expectant's life,
10,478 younger poſſeſſor's life,
7,333 elder poſſeſſor's;
And (86—43=) 43 will be the expectant's complement
(86—54=) 32 younger poſſeſſor's,
(86—66=) 20 elder poſſeſſor's;
Alſo (1+,04=) 1,04 the rate.

Now (32—1=) 31 being divided by (6×1,04=) 6,24 will quote 4,9689, which, being taken from 10,478, the remainder, will be 5,510; which, being multiplied by (32+1=) 33 will give 181,83, for the product to be reſerved.

Alſo (20—1=) 19 being divided by (6×1,04=) 6,24, will quote 3,045, which, ſubtracted from 7,333, will give 4,288, for the remainder to be reſerved.

From (2×43=) 86, ſubtract 20 & 1, the remainder will be 65; which, being multiplied by (20+1=) 21, will produce 1365, and this product, being multiplied by 4,288 (the remainder above reſerved) produces 5853,12; which, being divided by (2×32=) 64, will quote 91,455.

If the laſt found quotient 91,455, be ſubtracted from 181,83 (the product above reſerved) and the remainder, 90,375, be divided by (2×43=) 86, the quotient will be 1,051.

Laſtly; if, to (12,683—10,478=) 2,205, the above quotient 1,051, be added, their ſum, 3,256, will be the value of the reverſion required.

CASE 2. When the poſſeſſors are of equal ages, and both elder than the expectant.

The ſolution of this caſe may be deduc'd from the ſolution of caſe 1, by writing n for m; and $\frac{1}{rn}$ for $\frac{1}{rm}$; as follows,

$$\overline{\frac{m}{rt}-\overline{\frac{t-n\times 2}{rn}}}\times\frac{\mathcal{Q}}{nt}+t+\overline{\frac{3n-t}{rn}}\times\frac{\mathcal{Q}}{nnt}-\frac{1-p}{nnt}S$$

An approximation to this value may, alſo, be deduc'd from the approximation given in caſe 1, by writing n for m; and N for M, *viz.*

$$F-N+\frac{1}{2t}\times\overline{\overline{n+1}\times\overline{N-\frac{n-1}{6r}}-\overline{n+1}\times\frac{2t-n-1}{2n}\times\overline{\overline{\left(N-\frac{n-1}{6r}\right.}}}$$

Or, $F-N+\frac{1}{2t}\times\overline{n+1}\times\overline{N-\frac{n-1}{6r}}\times\overline{1-\frac{2t-n-1}{2n}}$:

But $1-\frac{2t-n-1}{2n}=\left(\frac{2n-2t+n+1}{2n}=\right)\frac{3n+1-2t}{2n}$;

Therefore the value of the reverſion will become

$$F-N+\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}\times\overline{3n+1-2t}}{4nt}$$

Which, expreſſed in words at length, follows :

The rule, for finding the preſent value of the reverſion of an annuity, to continue during a life, of a given age, after the deceaſe of two other perſons of equal ages, and both elder than the former; having the value of the ſingle lives given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which each perſon wants of 86, *be called their complements of life, and let one pound, and its intereſt for one year, be called the rate.*

From the complement of the life of one of the poſſeſſors, ſubtract one, and divide the remainder by ſix times the rate, or find the quotient in the laſt table; ſubtract the quotient from the value of one of the poſſeſſors life, reſerving the remainder.

Find the difference between, a number greater by one than three times one of the poſſeſſor's complement, and twice the expectant's complement; multiply this difference by the poſſeſſor's complement more one; *and multiply this product by the remainder, above reſerved; divide this laſt product by ſix times the product of the two different complements.*

Then, if the number, which is greater by one than the poſſeſſor's complement exceeds twice the expectant's complement, add this quotient to, or if that number be leſs than twice the expectant's complement, ſubtract this quotient from, the difference of the values of the ſingle lives of the expectant and one of the poſſeſſors, ſo ſhall their ſum, or remainder, be the value of the reverſion required.

EXAMPLE.

If the expectant be 43 years old, the two poſſeſſors each 66, and intereſt at four *per Cent*.

Then 12,683 will be the value of the life of the expectant,
7,333 each poſſeſſor;
And (86−43=) 43 will be the expectant's complement,
(86—66=) 20 the poſſeſſor's;
Alſo (1+,04=) 1,04 the rate.

Now if (20—1=) 19, be divided by (6,×1,04=)6,24, the quotient will be 3,045, which, ſubtracted from 7,333, will leave 4,288, for the remainder to be reſerved.

The number which is greater by one than (3×20=)60 will be 61; and (2×43=) 86; the difference of which is 25; which, being multiplied by (20+1=) 21, will produce 525; and this product, being multiplied by 4,288, (the remainder above reſerved) will produce 2251,2; which laſt product, being divided by (4×20×43=) 3440, will quote 0,654.

Now (becauſe 86 exceeded 61) the quotient 0,654, is to be ſubtracted from (12,683—7,333=) 5,350, and the remainder, 4,696, will be the value of the reverſion required.

CASE 3. When the expectant is elder than the poſſeſſors, and they are of unequal ages.

Let n be the complement of the expectant's life, and t and m thoſe of the poſſeſſor's.

Then will the value of an annuity on the longeſt of the two poſſeſſors lives be

$$P+\overline{\frac{1}{tr^t}+\frac{1}{tr^m}}\times Q-\overline{\frac{1}{tr^m}-\frac{1}{tmr^m}}\times R;$$

which, being taken from the value of the longeſt of the three lives, will leave

$$\frac{n}{mtr^n}\times Q+\overline{\frac{1}{tm}+\frac{2}{mtr^n}}\times R-\frac{1-p}{nmt}S,$$

$$\text{Or, }\frac{1}{mt}\times\overline{np\,Q+\overline{1+2p}\times R}-\frac{1-p}{n}S.$$

Now the approximation, to the longeſt of the poſſeſſors lives, will be

$\overline{F+M}-\frac{m-1}{6r}\times\frac{m+1}{2t}$; which, being taken from the approximation, to the value of the longeſt of the three lives, there will remain,

$\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^2}{4mt}$, for the value of the reverſion required.

Which, expreſſed in words at length, follows:

The rule for finding the preſent value of the reverſion of an annuity, to continue during the life of a perſon of a given age, after the deceaſe of two perſons of given ages, both younger than the former, having the value of the expectant's life given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which each perſon wants of 86, *be called their complements of life; and let one pound, and its intereſt for one year, be called the rate.*

From the complement of the expectant's life, ſubtract one; and divide the remainder by ſix times the rate, or find the quotient by the laſt table; ſubtract the quotient from the value of the expectant's life, reſerving the remainder.

To the complement of the expectant's life, add one; multiply that ſum by itſelf; and multiply the product by the remainder above reſerved.

Divide the product laſt found, by four times the product of the two poſſeſſors complements; ſo ſhall the quotient be the value of the reverſion required.

EXAMPLE.

If the poſſeſſors are ſeverally 43 and 54 years old, the expectant 66, and intereſt at four *per Cent.*

Then 7,333 will be the value of the expectant's life,
(86—66=) 20 the complement thereof,
(86—54=) 32 } the complements of the two
(86—43=) 43 } poſſeſſors;
And (1+,04=) 1,04, the rate;

Now if (20—1=) 19 be divided by (6×1,04=)6,24, the quotient will be 3,045, which, being ſubtracted from 7,333, will leave 4,288, for the remainder to be reſerved.

And if (20+1=) 21, be multiplied by 21, the product will be 441; which, being alſo multiplied by 4,288 (the remainder above reſerved) will produce 1891,008.

Alſo, if 1891,008 be divided by (4×43×32=) 5504, the quotient 0,343, will be the value of the annuity required.

CASE 4. When the poſſeſſors are of equal ages, and both younger than the expectant.

This caſe may be performed by the rules given for the laſt caſe; the whole difference being in the laſt diviſor; which, when the poſſeſſors ages are equal, will, in the firſt rule, be *mm*, inſtead of *mt*; and in the other, 4*mm*, inſtead of 4*mt*.

CASE 5. When the poſſeſſors are one elder, and the other younger, than the expectant.

Let *m* be the complement of the expectant's life, and *t* and *n* thoſe of the poſſeſſors.

Then the value of the annuity on the longeſt of the two poſſeſſors lives will be

$$P + \overline{\frac{1}{tr^t} + \frac{1}{tr^n}} \times Q - \overline{\frac{1}{nt} - \frac{1}{ntr^n}} \times R;$$

which, being ſubtracted from the value of an annuity, on the longeſt of the three lives, there will remain

$$\overline{\frac{1}{tr^m} - \overline{\frac{1}{t} - \frac{n}{mt}} \times \frac{1}{r^n}} \times Q + \overline{\left(\frac{1}{nt} + \frac{1}{mtr^m} + \frac{2}{mt} - \frac{1}{nt} \times \frac{1}{r^n}\right)} \times R - \frac{1-p}{nmt}S, \text{ Or}$$

$$\overline{\frac{m}{r^m} - \frac{m-n}{r^n}} \times \frac{Q}{mt} + \overline{m + \frac{n}{r^m} + \frac{2n-m}{r^n}} \times \frac{R}{nmt} - \left(\frac{\overline{1-p} \times S}{nmt}\right.$$

the value of the reverſion required.

Now the approximation to the value of an annuity for the longeſt of the two lives of the poſſeſſors will be

$$F - N - \overline{\frac{n-1}{6r}} \times \frac{n+1}{2t};$$ which, being taken from the value of the longeſt of the three lives, will leave

$$\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^{2}}{4mt}-\overline{N-\frac{n-1}{6r}}\times\frac{n+1}{2t}+\overline{M-\frac{m-1}{2m}}\times\frac{m+1}{2t}:$$

Now $\frac{\overline{n+1}^{2}}{4mt}=\frac{n+1}{2t}\times\frac{n+1}{2m}$; and $\frac{n+1}{2m}-1=-\frac{2m-n-1}{2m}$;

whence the value of the reverſion will be

$$\overline{M-\frac{m-1}{6r}}\times\frac{m+1}{2t}-\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}\times\overline{2m-n-1}}{4mt}; \text{ or}$$

$$\frac{1}{2t}\times\overline{\overline{M-\frac{m-1}{6r}}\times\overline{m+1}-\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}\times\overline{2m-n-1}}{2m}}.$$

Which, in words at length, is as follows:

The rule to find the preſent value of the reverſion of an annuity, to continue during the life of a perſon of a given age, after the deceaſe of two other perſons, one of which is elder, and the other younger, than the expectant; having the values of the ſingle lives of the expectant, and eldeſt poſſeſſor, given; allowing compound intereſt at a given rate; and ſuppoſing the decrements of life to be equal.

Let the number of years, which each of the perſons want of 86, *be called their complements of life; and let one pound, and its intereſt for one year, be called the rate.*

From the complement of the expectant's life, ſubtract one; and divide the remainder by ſix times the rate, or find the quotient in the laſt table.

Subtract this quotient from the value of the expectant's life; and multiply the remainder by the expectant's complement more one, *reſerving the product.*

From the elder poſſeſſor's complement ſubtract one, and divide the remainder by ſix times the rate, or find this quotient in the laſt table; ſubtract the quotient from the value of the elder poſſeſſor's life, reſerving the remainder.

From twice the expectant's complement, ſubtract the complement of the elder poſſeſſor, and one; multiply the remainder by the elder poſſeſſor's complement more one; *and multiply that product by the remainder above reſerved; dividing the laſt product by twice the expectant's complement.*

From the product above reſerved, take the laſt found quotient; and divide the remainder by twice the complement of the younger poſſeſſor, ſo ſhall the quotient be the value of the reverſion required.

EXAMPLE.

If the expectant be 54 years old, the poſſeſſors ſeverally 43, and 66; and intereſt at four *per Cent.*

Then 10,478 will be the value of the expectant's life,
7,333, that of the elder poſſeſſor
(86—66=) 20 the complement of the elder poſſeſſor
(86—54=) 32 expectant,
(86—43=) 43 younger poſſeſſor;
And (1+,04=) 1,04 the rate.

Now if (32—1=) 31 be divided by (6×1,04=) 6,24, the quotient will be 4,968; which, being ſubtracted from 10,478; and the remainder 5,510, be multiplied by (32+1=) 33, the product will be 181,83, which is to be reſerved.

And if (20—1=) 19, be divided by 6,24; and the quotient 3,045, be ſubtracted from 7,333, the remainder 4,288, is to be reſerved.

Again $(2\times32-20-1=)$ 43, being multiplied by $(20+1=)$ 21, the product will be 903; which, being multiplied by 4,288 (the remainder above reſerved) will produce 3872,0; and this divided by $(2\times32=)$ 64, will quote 60,5.

Laſtly, if from 181,83 (the product above reſerved) this quotient, 605, be ſubtracted; and the remainder 121,33, be divided by $2\times43=)$ 86, the quotient 1,411, will be the value of the reverſion required.

CASE 6. When the elder poſſeſſor and expectant are of the ſame age.

The ſolution of this caſe may be derived from the laſt; by writing n for m; and r^n for r^m; whence the value of the reverſion will become

$$\frac{Q}{tr^n}+\overline{n+\frac{2n}{r^n}}+\frac{R}{nnt}-\frac{1-p}{nnt}S,\ \text{Or},$$

$$\frac{Qp}{t}+\overline{1+2p}\times\frac{R}{nt}-\frac{1-p}{nnt}S;$$

And the approximation thereto will, by writing n for m; and N for M, become

$$\frac{1}{2t}\times\overline{\overline{N-\frac{n-1}{6r}}\times\overline{n+1}-\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}\times\overline{2n-n-1}}{2n}};$$

that is $\frac{1}{2t}\times\overline{\overline{N-\frac{n-1}{6r}}\times\overline{n+1}\times\overline{1-\frac{n-1}{2n}}}$

But $1-\frac{n-1}{2n}=\left(\frac{2n-n+1}{2n}=\right)\frac{n+1}{2n}$; therefore the approximation will become

$\frac{1}{2t}\times\overline{N-\frac{n-1}{6r}}\times\overline{\frac{n+1}{2n}}^2$, or $\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^2}{4nt}$; which may be expreſſed by the words given in the rule to caſe 3.

Case 7. When the younger poſſeſſor and expectant are of the ſame age.

This may be alſo deduced from caſe the 5th, by writing m for t; when the value of the reverſion will be

$$\overline{\frac{m}{r^m}-\frac{m-n}{r^n}}\times\frac{Q}{mn}+\overline{m+\frac{n}{r^m}+\frac{2n-m}{r^n}}\times\frac{R}{nmm}-\frac{1-p}{nmm}S;$$

and the approximation thereto

$$\frac{1}{2m}\times\overline{M-\frac{m-1}{6r}\times\overline{m+1}-N-\frac{n-1}{6r}\times\frac{\overline{n+1}\times\overline{2m-n-1}}{2m}};$$

which may be expreſſed, in the ſame words, as the rule in caſe 5.

Case 8. When the expectant and the poſſeſſors are of the ſame age.

The ſolution of this caſe may be deduced from any of the former (ſuppoſe the 6th) by writing n for t, whence the value of the reverſion will become

$$\frac{Qp}{n}+\overline{1+2p}\times\frac{R}{nn}-\frac{1-p}{nnn}S;$$

and the approximation thereto, $\overline{N-\frac{n-1}{6r}}\times\frac{\overline{n+1}^2}{4nn}$:

But $(\overline{n+1}^2=)\ nn+2n+1=(nn+2n=)\ n\times\overline{n+2}$ nearly;
Therefore, for the approximation may be wrote,

$$\overline{N-\frac{n-1}{6r}}\times\frac{n\times\overline{n+2}}{4nn},\text{ or }\overline{N-\frac{n-1}{6r}}\times\frac{n+2}{4n}.$$

Which, expreſſed in words at length, follows:

The rule for finding the value of the reverſion of an annuity, which is to continue during a life of a given age, after the deceaſe of two other perſons of the ſame age; having the value of the ſingle life given, allowing compound intereſt at a given rate, and ſuppoſing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, *be called the complement of life; and let one pound, and its intereſt for one year, be called the rate.*

From the complement of life, ſubtract one; and divide the remainder by ſix times the rate, or find this quotient in the laſt table.

Subtract this quotient from the value of the ſingle life; multiply the remainder by the complement more two; *and divide the product by four times the complement; ſo ſhall the quotient be the value of the reverſion required.*

EXAMPLE.

If the three perſons be each 66 years of age, and the rate of intereſt four *per Cent.*

Then 7,333 will be the value of the ſingle life.

(86— 66=) 20, will be the complement of life.

And (1+,04=) 1,04, the rate.

If (20— 1 =) 19, be divided by (6×1,04=) 6,24, the quotient will be 3,045; which, being ſubtracted from 7,333, leaves 4,288; this being multiplied by (20+2=) 22, produces 94,34; which, divided by (4×20=) 80, gives 1,179, for the value of the reverſion required.

QUESTION CII.

Suppoſing the decrements of life to be equal, it is required to find the value of the reverſion of an annuity, which is to continue during the life of a perſon of a given age, after the longeſt liver of three perſons of the ſame age.

SOLU-

SOLUTION.

Let the complement of life be repreſented by n; then will the probability of the failing of the three poſſeſſors lives, in the firſt, ſecond, third, &c. years, be

$$1-\overline{\frac{n-1}{n}}^{3},\ 1-\overline{\frac{n-2}{n}}^{3},\ 1-\overline{\frac{n-3}{n}}^{3}\ \&c.$$

$$\text{Or } 1-\frac{\overline{n-1}\times3}{n}+\frac{\overline{n-1}^{2}\times3}{n^{2}}-\frac{\overline{n-1}^{3}}{n^{3}},$$

$$1-\frac{\overline{n-2}\times3}{n}+\frac{\overline{n-2}^{2}\times3}{nn}-\frac{\overline{n-2}^{3}}{n^{3}},$$

$$1-\frac{\overline{n-3}\times3}{n}+\frac{\overline{n-3}^{2}\times3}{nn}-\frac{\overline{n-3}^{3}}{n^{3}},\ \&c.$$

which being ſeverally multiplied by $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c. the probabilities of the expectant's living one, two, three, &c. years, and the preſent worths of thoſe probabilities taken, will produce

$$\begin{array}{cccc}
\frac{n-1}{nr}-\frac{\overline{n-1}^{2}\times3}{nnr}&+\frac{\overline{n-1}^{3}\times3}{nnnr}&-\frac{\overline{n-1}^{4}}{n^{4}r}\\
\frac{n-2}{nr^{2}}-\frac{\overline{n-2}^{2}\times3}{nnr^{2}}&+\frac{\overline{n-2}^{3}\times3}{nnnr^{2}}&-\frac{\overline{n-2}^{4}}{n^{4}r^{2}}\\
\frac{n-3}{nr^{3}}-\frac{\overline{n-3}^{2}\times3}{nnr^{3}}&+\frac{\overline{n-3}^{3}\times3}{nnnr^{3}}&-\frac{\overline{n-3}^{4}}{n^{4}r^{3}}\\
\&c.\quad\&c.&\&c.&\&c.
\end{array}$$

for the value of the reverſion required.

$$\text{Now}\left\{\begin{array}{l} \frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}\ \&c. \\[6pt] \frac{\overline{n-1}^{\,2}}{n^2r}+\frac{\overline{n-2}^{\,2}}{n^2r^2}+\frac{\overline{n-3}^{\,2}}{n^2r^3}\ \&c. \\[6pt] \frac{\overline{n-1}^{\,3}}{n^3r}+\frac{\overline{n-2}^{\,3}}{n^3r^2}+\frac{\overline{n-3}^{\,3}}{n^3r^3}\ \&c. \\[6pt] \frac{\overline{n-1}^{\,4}}{n^4r}+\frac{\overline{n-2}^{\,4}}{n^4r^2}+\frac{\overline{n-3}^{\,4}}{n^4r^3}\ \&c. \end{array}\right\}\begin{array}{l}\text{in the value of the joint}\\ \text{lives of}\end{array}\left\{\begin{array}{c}1\\2\\3\\4\end{array}\right\}\text{perſons of the given age by}\left\{\begin{array}{c}56\\65\\72\\75\end{array}\right.$$

Therefore the value of the reverſion required will be, the ſingle life, leſs thrice the value of two equal joint lives, more thrice the value of three equal joint lives, more the value of four equal joint lives;

That is per corol. to queſt. 75.

$$P-\frac{1-p}{n}Q,$$

$$-3P+\frac{3\times 2}{n}Q-\frac{3\times\overline{1-p}}{nn}R,$$

$$+3P-\frac{3\times 3}{n}Q+\frac{3\times 3}{nn}R-\frac{3\times\overline{1-p}}{n^3}S,$$

$$-P+\frac{4}{n}Q-\frac{6}{nn}R+\frac{4}{n^3}S-\frac{1-p}{n^4}T;$$

The ſum of all which, *viz.*

$$\frac{p}{n}Q+\frac{3p}{nn}R+\frac{1+3p}{n^3}S-\frac{1-p}{n^4}T,$$ will be the value of the reverſion required.

QUESTION

QUESTION CIII.

Suppoſing the decrements of life to be equal; it is required to find the value of the reverſion of an annuity, which is to continue during the life of a perſon of a given age, after the longeſt liver of four perſons of the ſame age.

SOLUTION.

By reaſoning in the ſame manner as in the laſt queſtion, it will appear, that the value of the required reverſion will be, the value of the ſingle life, leſs four times the value of two equal joint lives, more ſix times the value of three equal joint lives, more four times the value of four equal joint lives, leſs the value of five equal joint lives.

That is, per corol. to queſt. 75.

$$P-\frac{1-p}{n}Q,$$

$$-4P+\frac{4\times 2}{n}Q-\frac{4\times\overline{1-p}}{nn}R,$$

$$+6P-\frac{6\times 3}{n}Q+\frac{6\times 3}{nn}R-\frac{6\times\overline{1-p}}{n^3}S,$$

$$-4P+\frac{4\times 4}{n}Q-\frac{4\times 6}{nn}R+\frac{4\times 4}{n^3}S-\frac{4\times\overline{1-p}}{n^4}T,$$

$$+P-\frac{5}{n}Q+\frac{10}{nn}R-\frac{10}{n^3}S+\frac{5}{n^4}T-\frac{1-p}{n^5}V;$$

The ſum of all which, *viz.*

$$\frac{p}{n}Q+\frac{4p}{nn}R+\frac{6p}{n^3}S+\frac{1+4p}{n^4}T-\frac{1-p}{n^5}V,$$

will be the value of the reverſion required.

COROL.

COROL. I.

Hence, the value of the reverſion of one life, after the longeſt of m equal lives, will be,

$$\frac{p}{n}Q+\frac{mp}{nn}R+\frac{m\cdot\overline{m-1}}{nn:2n}S+\frac{m\cdot\overline{m-1}\cdot\overline{m-2}}{nn.2n\cdot 3n}T\ (m-1)$$

$$\left(+\frac{1+mp}{n^{m}}Y-\frac{1-p}{n^{m+1}}Z;\right.$$

Where Y, and Z, are the $\overline{m+1}$th and $\overline{m+2}$th terms in the ſeries of factors, mentioned in queſt. 20, and all the terms in the expreſſion are affirmative, but the laſt.

COROL. II.

By comparing the above expreſſion, with thoſe given for the longeſt of any number of lives (in corol. to queſt. 86) it will appear, that the reverſion of one life after the longeſt of m equal lives, will be the difference between the longeſt of $\overline{m+1}$ equal lives, and the longeſt of m equal lives.

QUESTION CIV.

A table of obſervations, deduced from the bills of mortality of any place, being given; to find the complement of that life, whoſe value (being computed according to the hypotheſis of equal decrements) ſhall be nearly equal to the value of the given life, computed from the given table of obſervations.

Let n be the required complement of life; then the probabilities of the life's continuing, to the extremity of old age, will (by arguing as before) be

$$\frac{n-1}{n}+\frac{n-2}{2}+\frac{n-3}{n}+\frac{n-4}{n}\,(n)$$

which being an arithmetical progreſſion, whoſe greateſt term is $\frac{n-1}{n}$, leaſt term nothing, and number of terms n, the ſum thereof, will be $\overline{\frac{n-1}{n}+0}\times\frac{n}{2}$ (by queſt. 7, part II. vol. I. Or $\left(\frac{n-1}{n}\times\frac{n}{2}=\right)\frac{n-1}{2}$

This expreſſion ſhould (in order to make the lives of the ſame value) be equal to the ſum of a like number of terms, taken from the given table of obſervations, *viz.* thoſe which expreſs the probabilities of the given life's continuing to the age of 86, ſuppoſed as before to be the extremity of old age; the numerators of which terms will, if the table of obſervations be diſpoſed in the manner of thoſe given in page 157 and 159, be found in the ſecond column thereof, and may be added together upon the face of the table; their common denominator being the number, which, in the ſame column, ſtands againſt the given age.

Now if s repreſent the ſum of thoſe numerators, and a their common denominator, then the following equation will ariſe

$$\frac{n-1}{2}=\frac{s}{a};$$

Whence $n-1=\frac{2s}{a}$, and $n=\frac{2s}{a}+1$.

Upon which principle the following table is compoſed, ſhewing thoſe complements, which correſpond to the ſeveral ages therein mentioned, according to the table of obſervations, deduced from the bills of mortality of *London.*

Age

Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.
8	71,5	20	57,8	32	45.5	44	36,2	56	27,3	68	18,3
9	70,7	21	56,7	33	44.7	45	35.5	57	26,5	69	17,7
10	69,8	22	55,6	34	43,8	46	34,8	58	25,8	70	17,2
11	68,7	23	54,5	35	42,9	47	34,1	59	25,1	71	16,4
12	67,6	24	53,3	36	42,1	48	33,3	60	24,4	72	15,7
13	66.4	25	52,3	37	41,2	49	32,5	61	23,7	73	15,1
14	65,2	26	51,3	38	40,4	50	31,8	62	22,8	74	14.5
15	64,0	27	50,3	39	39.5	51	31,0	63	22,0	75	13,8
16	62,8	28	49,2	40	38,8	52	30,3	64	21,2	76	13,0
17	61,6	29	48,2	41	38,2	53	29,6	65	20,5	77	11,9
18	60,3	30	47,3	42	37.5	54	28,9	66	19,7	78	10,9
19	59,0	31	46,5	43	36,9	55	28,1	67	19,0	79	9,8

Hence (if a table of the above kind be computed to correſpond with every table of obſervations, that now are, or may be hereafter obtained) the values of annuities on ſingle and combined lives, with their reverſions, may be obtained nearly, by taking their complements, or the whole numbers neareſt thereto, from ſuch tables, and applying them to the proper ſolutions, inſtead of the differences between the given ages and 86.

EXAMPLE I.

If the values of the ſingle lives of 10, and 70, at four *per Cent*. be required; according to the *London* obſervations.

The reſpective tabular complements of thoſe ages are 70, and 17;

The ages, which (when the decrements of life are ſuppoſed to be equal) correſpond to thoſe complements are 16, and 69;

The values of which are (*per* table page 169, 171) 16, 3, and 6, 4:

And the values of theſe lives, according to Mr. *Simpſon*'s tables, are 16,4, and 6,5.

EXAMPLE II.

Let the value of the joint lives of the ages 10 and 31, at four *per Cent.* according to the *London* obſervations (which was found, by queſt. 68, to be 10,8) be required by this method, applied to the approximation in queſt. 64.

The value of the ſingle life of 31 is, by Mr. *Simpſon*'s tables, 12,9; and the tabular complements of the given ages give 70, and 47.

Now if from 12,9 be taken $\left(\frac{n-1}{6r}=\frac{46}{6,24}=\right)$ 7,4, the remainder will be 5,5; this being multiplied by $\left(\frac{n+1}{2m}=\frac{48}{140}=\right)\frac{12}{35}$, produces 1,9; which, taken from 12,9, leaves 11,0, for the value required.

The method above given, for finding the complement of a life, according to any table of obſervations, may be extended to the finding the expectations of ſuch lives.

The *expectation of life* is that time, which a perſon, of a given age, may juſtly expect to continue in being.

QUESTION CV.

What is the expectation of a ſingle life of a given age?

By the proceſs in queſt. 104, when the decrements of life are equal, the ſum of the probabilities, which a given life, whoſe complement is n, has of continuing to the extremity of old age, will be $\frac{n-1}{2}$.

This expreſſion will not be the whole expectation of life; becauſe the ſeries, $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}$ &c. whoſe ſum it is, conſiſts (only) of the probabilities of the given life's ſurviving the firſt, ſecond, third, &c. years, without any allowance for the time, which the life may continue after the end of ſuch year, even though it ſhould fail before the expiration of the next.

Now when a life has attained the beginning of that year, in which it may be ſuppoſed to fail; it will be an equal chance, whether it fails in the firſt half year, or the latter half year thereof; and therefore the probability of ſurviving half of the year ſhould be added to each of the terms of that ſeries.

Thus, in the firſt year, the probability of ſurviving to the end thereof is $\frac{n-1}{n}$, and that of failing $\frac{1}{n}$; therefore, if $\frac{1}{2n}$, the half of the latter, be added to $\frac{n-1}{n}$, the former; the ſum $\left(\frac{n-1}{n}+\frac{1}{2n}=\right)\frac{2n-1}{2n}$, will be the expectation of life for that year; and by arguing in the ſame manner, the expectation of life, for the ſecond, third, fourth, &c. years, will be $\frac{2n-3}{2n}$, $\frac{2n-5}{2n}$, $\frac{2n-7}{2n}$, &c. the nth term of which, *viz.* $\frac{2n-\overline{2n-1}}{2n}$, will be $\frac{1}{2n}$: This being an arithmetical progreſſion, whoſe greateſt term is $\frac{2n-1}{2n}$; leaſt term $\frac{1}{2n}$, and number of terms n, therefore $\overline{\frac{2n-1}{2n}+\frac{1}{2n}}\times\frac{n}{2}$, Or $\left(\frac{2n}{2n}\times\frac{n}{2}=\right)\frac{n}{2}$ will be the whole expectation of the given life.

Hence the expectation of a ſingle life, according to any table of obſervations, will be nearly equal to half the complement ſound by queſt. 104.

QUESTION CVI.

What is the expectation of two joint lives, of given ages?

Let the complement of the elder life be n, and that of the younger m; then, by arguing as before, the expectation of their joint continuance, for one, two, three, &c. years will be

$$\frac{2n-1}{2n}\times\frac{2m-1}{2m},\ \frac{2n-3}{2n}\times\frac{2m-3}{2m},\ \frac{2n-5}{2n}\times\frac{2m-5}{2m}\ \&c.$$

and the ſum of n terms of that ſeries will be the expectation of the joint lives.

Now the terms of this ſeries are the products of the two arithmetical progreſſions, $\frac{2n-1}{2n}, \frac{2n-3}{2n}, \frac{2n-5}{2n}\ (n)$, and $\frac{2m-1}{2m}, \frac{2m-3}{2m}, \frac{2m-5}{2m}\ (n)$; whoſe common differences are $\frac{1}{n}$ and $\frac{1}{m}$; the ſum of the firſt being $\frac{n}{2}$: And ſince the greateſt term of the ſecond is $\frac{2m-1}{2m}$, the leaſt $\frac{2m-\overline{2n-1}}{2m}$, and number of terms n; the ſum thereof will be

$$\overline{\left(\frac{2m-1}{2m}+\frac{2m-2n+1}{2m}\times\frac{n}{2}=\right)}\ \frac{4m-2n}{4m}\times n$$

$$\text{Or } \overline{1-\frac{n}{2m}}\times n$$

Therefore, by queſt. 21, the ſum of n terms of the ſeries of products will be $\frac{\frac{n}{2}\times\overline{1-\frac{n}{2m}}\times n}{n}+\frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot2\cdot3}\times$

$$\left(\frac{1}{n}\times\frac{1}{m};\right.$$

$$\text{Or } \frac{n}{2}\times\overline{1-\frac{n}{2m}}+\frac{\overline{n+1}\times\overline{n-1}}{4\cdot3m};$$

$$\text{Or } \frac{n}{2}-\frac{nn}{4m}+\frac{nn-1}{12m}$$

Now, becauſe $\frac{nn-1}{12m}$ differs from $\frac{nn}{12m}$ only by $\frac{1}{12m}$ (a quantity too ſmall to affect this calculation) therefore the above may be wrote $\left(\frac{n}{2}-\frac{nn}{4m}+\frac{nn}{12m}=\right)\frac{n}{2}-\frac{nn}{6m}$; which is the expectation of the two joint lives.

COROL.

Hence alſo, if the two ages be equal, the expectation of their joint lives will be $\left(\frac{n}{2}-\frac{nn}{6n}=\frac{n}{2}-\frac{n}{6}=\right)\frac{n}{3}$.

QUESTION CVII.

What is the expectation of three joint lives, of given ages?

Let the complements of the eldeſt, ſecond, and youngeſt, be ſeverally denoted by n, m, and t; then will the expectation required (by arguing as before) be

$$\frac{2n-1}{2n}\times\frac{2m-1}{2m}\times\frac{2t-1}{2t}+\frac{2n-3}{2n}\times\frac{2m-3}{2m}\times\frac{2t-3}{2t}\ (n):$$

Now ſince the terms of this ſeries are the products of the terms of three arithmetical progreſſions, whoſe ſums are $\frac{n}{2}$, $\overline{1-\frac{n}{2m}}\times n$, and $\overline{1-\frac{n}{2t}}\times n$; their common differences $\frac{1}{n}, \frac{1}{m}, \frac{1}{t}$; and the number of their terms n; therefore the ſum of this ſeries of products will (by queſt. 22) be

$$\frac{n}{2}\times\overline{1-\frac{n}{2m}}\times\overline{1-\frac{n}{2t}}+\frac{\overline{n+1}\cdot n\cdot\overline{n-1}}{2\cdot 2.3}$$

$$\times\overline{\frac{2n-1}{2n\cdot mt}+\frac{2m-1}{2m\cdot nt}+\frac{2t-1}{2t\cdot nm}}-\frac{\overline{n+1}\cdot n\cdot\overline{n-1}^{2}}{2\cdot 2\cdot 2}\times\frac{1}{nmt};$$

$$\text{Or }\frac{n}{2}\times\overline{1-\frac{n}{2m}}\times\overline{1-\frac{n}{2t}}+\frac{\overline{n+1}\times\overline{n-1}\times\overline{2n+2m+2t-3}}{2\cdot 2\cdot 2\cdot 3\cdot mt}$$

$$\Big(-\frac{\overline{n+1}\cdot\overline{n-1}^{2}}{2\cdot 2\cdot 2\cdot mt};$$

$$\text{Or }\frac{n}{2}\times\frac{2m-n}{2m}\times\frac{2t-n}{2t}+\frac{\overline{2n+2m+2t-3}}{3}\,\overline{-n-1}$$

$$\Big(\times\frac{\overline{n+1}\cdot\overline{n-1}}{2\cdot 2\cdot 2mt};$$

Which, being reduced, and nn wrote for $\overline{n+1}\times\overline{n-1}$, as in the laſt queſtion, will give

$$\frac{12nmt-4n^2t-4n^2m+2n^3}{24mt}, \text{ Or}$$

$\frac{n}{2}-\frac{n^2}{6}\times\overline{\frac{1}{m}+\frac{1}{t}}+\frac{n^3}{12mt}$ for the expectation of three joint lives.

COROL. I.

Hence, if the lives be of equal ages, their expectation will be$\left(\frac{n}{2}-\frac{n^2}{6n}-\frac{n^2}{6n}+\frac{n^3}{12n^2}=\frac{n}{2}-\frac{2n}{6}+\frac{n}{12}=\frac{6n-4n+n}{12}=\right)\frac{n}{4}$

COROL. II.

Hence alſo the manner of finding the expectation of any number of joint lives is ſufficiently evident, and may be expreſſed in the words uſed by Mr. *De Moivre*, viz.

The rule to find the expectation of any number of joint lives.

“ *Take* $\frac{1}{2}$ *of the ſhorteſt complement :*

“ *Take* $\frac{1}{6}$ *part of the ſquare of the ſhorteſt, which divide, ſucceſſively, by all the other complements ; then add all the quotients together :*

“ *Take* $\frac{1}{12}$ *part of the cube of the ſhorteſt complement ; which divide, ſucceſſively, by the products of all the other complements, taken two and two :*

“ *Take* $\frac{1}{20}$ *part of the biquadrate of the ſhorteſt complement ; which divide, ſucceſſively, by the products of all the other complements, taken three and three, and ſo on.*

“ *Then from the reſult of the firſt operation, ſubtract the reſult of the ſecond, to the remainder add the reſult of the third, from the ſum ſubtract the reſult of the fourth, and ſo on.*

“ *The laſt quantity, remaining after theſe alternate ſubtractions and additions, will be the thing required.*

“ N. B. *The diviſors* 2, 6, 12, 20, *&c. are the products of* 1 *by* 2, *of* 2 *by* 3, *of* 3 *by* 4, *of* 4 *by* 5, *&c.*

COROL.

“ *But if all the lives be equal, add unity to the number of lives, and divide their complement by that number, thus increaſed by unity ; and the quotient will expreſs the time due to their joint continuance.*”

QUESTION CVIII.

What is the expectation of the longeſt of two lives, of given ages ?

From the ſum of the expectations of the ſingle lives - - - - - - - - - - . $\frac{n}{2}+\frac{m}{2}$:

Take the expectation of the two joint lives - $\frac{n}{2}-\frac{nn}{6m}$:

The remainder will be the expectation of the longeſt of thoſe lives - - - : - $\frac{m}{2}+\frac{nn}{6m}$.

COROL.

COROL.

Hence, if the lives be of equal ages, the expectation of the longeſt will be $\left(\frac{n}{2}+\frac{nn}{6n}=\frac{n}{2}+\frac{n}{6}=\right)\frac{2n}{3}$.

QUESTION CIX.

What is the expectation of the longeſt of three lives, of given ages?

To the ſum of the expectations of the three ſingle lives - $\frac{n}{2}+\frac{m}{2}+\frac{t}{2}$;

Add the expectation of the three joint lives - - - - $\frac{n}{2}-\frac{n^2}{6m}-\frac{n^2}{6t}+\frac{n^3}{12mt}$;

The ſum will be $n+\frac{m}{2}+\frac{t}{2}-\frac{n^2}{6m}-\frac{n^2}{6t}+\frac{n^3}{12mt}$

To the expectation of the two joint lives, whoſe complements are n and m $\frac{n}{2}-\frac{nn}{6m}$;

Add, that of the lives, whoſe complements are n and t, - - - - - - $\frac{n}{2}-\frac{nn}{6t}$;

And, that of the lives, whoſe complements are m and t, - - - - $\frac{m}{2}-\frac{mm}{6t}$;

The ſum will be $n+\frac{m}{2}-\frac{nn}{6m}-\frac{nn}{6t}-\frac{mm}{6t}$:

And the difference of theſe ſums, $\frac{t}{2}+\frac{mm}{6t}+\frac{n^3}{12mt}$, will be the expectation required.

COROL. I.

If the lives be of equal ages, the expectation of the longeſt will be

$$\left(\frac{n}{2}+\frac{nn}{6n}+\frac{n^3}{12nn}=\frac{n}{2}+\frac{n}{6}+\frac{n}{12}=\frac{9n}{12}=\right)\frac{3n}{4}.$$

COROL. II.

Hence the method of finding the expectation of the longeſt of any number of lives may be expreſſed in words at length as follows:

The rule for finding the expectation of the longeſt of any number of lives, of given ages.

Let the complement of the longeſt life be called the firſt complement; that of the next elder, the ſecond; that of the next elder, the third; and ſo on.

Take half the firſt complement; divide one ſixth part of the ſquare of the ſecond complement by the firſt complement; divide one twelfth part of the cube of the third complement by the products of the firſt and ſecond; divide one twentieth part of the fourth power of the fourth complement by the continual product of the three former complements; and ſo continue, dividing the one thirtieth part of the fifth power of the fifth complement, the one forty-ſecondth part of the ſixth power of the ſixth complement, &c. by the continual product of all the former complements; ſo ſhall the ſum of all theſe quotients be the expectation required.

But, if the lives are of equal ages, let their common complement be multiplied by the number of lives, and the product be divided by the number of lives more one; *ſo ſhall the quotient be the expectation required.*

If the expectation of any number of lives, according to a table of obſervations deduced from the bills of mortality of any place, be required; let the complements of thoſe lives be found by queſtion 104; *and then proceed as directed in the above-given rules.*

If any number is wanted that is beyond the reach of the ſucceeding table; the ſame may be found, by adding two, or more, of thoſe together; for inſtance, the product of $\frac{1}{6r}$, by 65, will be the ſum of the products thereof by 30 and 35, *viz.* (at four *per Cent.*) 4,8077+5,6089= 10,3166.

G.R.III. BR.MUS.

TABLE the laſt, being the multiples of $\frac{1}{6r}$ very uſeful in computing the approximations to the values of combined lives.

years	3 *per C.*	3½ *per C*	4 *per C.*	4½ *per C.*	5 *per C.*	6 *per C*
1	0,1618	0,1610	0,1603	0,1595	0,1587	0,1572
2	0,3236	0,3221	0,3205	0,3190	0,3175	0,3145
3	0,4854	0,4831	0,4808	0,4785	0,4762	0,4717
4	0,6472	0,6441	0,6410	0,6380	0,6349	0,6289
5	0,8091	0,8052	0,8013	0,7975	0,7936	0,7862
6	0,9709	0,9662	0,9615	0,9569	0,9524	0,9434
7	1,1327	1,1272	1,1218	1,1164	1,1111	1,1006
8	1,2945	1,2882	1,2820	1,2759	1,2698	1,2579
9	1,4563	1,4493	1,4423	1,4354	1,4286	1,4151
10	1,6181	1,6103	1,6026	1,5949	1,5873	1,5723
11	1,7799	1,7713	1,7628	1,7544	1,7460	1,7296
12	1,9417	1,9324	1,9231	1,9139	1,9048	1,8868
13	2,1036	2,0934	2,0833	2,0734	2,0635	2,0440
14	2,2654	2,2544	2,2436	2,2329	2,2222	2,2013
15	2,4272	2,4155	2,4038	2,3924	2,3809	2,3585
16	2,5890	2,5765	2,5641	2,5518	2,5397	2,5157
17	2,7508	2,7375	2,7244	2,7113	2,6984	2,6730
18	2,9126	2,8986	2,8846	2,8708	2,8571	2,8302
19	3,0744	3,0596	3,0449	3,0303	3,0159	2,9874
20	3,2362	3,2206	3,2051	3,1898	3,1746	3,1447
21	3,3981	3,3817	3,3654	3,3493	3,3333	3,3019
22	3,5599	3,5427	3,5256	3,5088	3,4921	3,4591
23	3,7217	3,7037	3,6059	3,6683	3,6508	3,6164
24	3,8835	3,8647	3,8461	3,8278	3,8095	3,7736
25	4,0453	4,0258	4,0064	3,9872	3,9682	3,9308
26	4,2071	4,1868	4,1667	4,1467	4,1270	4,0881
27	4,3689	4,3478	4,3269	4,3062	4,2857	4,2453
28	4,5307	4,5089	4,4872	4,4657	4,4444	4,4025
29	4,6925	4,6699	4,6474	4,6252	4,6032	4,5598
30	4,8544	4,8305	4,8077	4,7847	4,7619	4,7170
31	5,0162	4,9920	4,9679	4,9442	4,9206	4,8742
32	5,1780	5,1530	5,1282	5,1057	5,0794	5,0315
33	5,3398	5,3140	5,2884	5,2632	5,2381	5,1887
34	5,5016	5,4751	5,4487	5,4227	5,3968	5,3459
35	5,6634	5,6361	5,6089	5,5821	5,5555	5,5032

FINIS.